

# SLOSH DESIGN HANDBOOK I

*by*

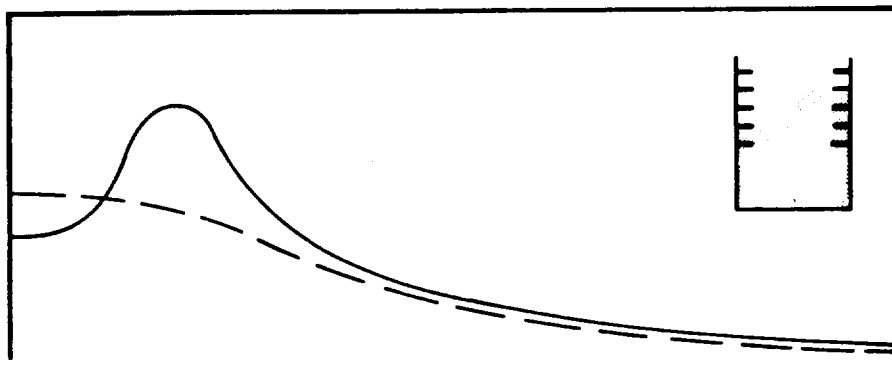
James R. Roberts

Eduardo R. Basurto

Pei-Ying Chen

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SLOSH DESIGN HANDBOOK I

By

James R. Roberts

Eduardo R. Basurto

Pei-Ying Chen

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### Foreword

This Handbook was prepared by Northrop Space Laboratories, Huntsville, Alabama, for the Dynamics and Analysis Section of the Aero-Astroynamics Laboratory of the George C. Marshall Space Flight Center. Its contents represent a partial fulfillment of the project since additional topics are forthcoming which, upon completion, will be integrated into the Handbook.



## Abstract

Design information related to the effects of propellant sloshing is presented for use in both control and structural problems. Both analytical and experimental results are given and all pertinent material is referenced. Graphs have been included, whenever possible, to expedite preliminary design calculations. The areas covered are: (1) linearized fluid theory, (2) equivalent mechanical model theory, (3) results of analytical studies of liquid oscillations in variously shaped containers when subjected to different types of excitation, i.e., boundary conditions, fluid velocity potentials, natural frequencies, liquid force and moment resultants and equivalent mechanical models, and (4) results of both analytical and experimental studies concerned with propellant slosh suppression, with particular emphasis on fixed-ring baffles.





## SLOSH DESIGN HANDBOOK

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# NOTATION

|             |   |  |
|-------------|---|--|
| $a$         | - | Radius of circular cylindrical tank<br>Outer radius of annular or annular-sector tank<br>Length of rectangular tank in x-direction |
| $a_o$       | - | Inner radius of ring baffle  |
| $b$         | - | Inner radius of annular or annular-sector tank   |
| $C_D$       | - | Local drag coefficient   |
| $c$         | - | Radial clearance between tank wall and ring baffle   |
| $\bar{c}$   | - | Effective damping coefficient  |
| $c_d$       | - | Damping coefficient of disc  |
| $c_n$       | - | Damping coefficient of $n^{th}$ slosh mass   |
| $c.g.$      | - | Center of gravity of undisturbed liquid  |
| $D$         | - | Baffle spacing<br>Tank diameter  |
| $d$         | - | Tank diameter<br>Depth of baffle below undisturbed free surface  |
| $d_s$       | - | Depth of baffle below undisturbed free surface   |
| $d^*$       | - | Distance of baffle above the undisturbed free surface,<br>$D - d$  |
| $F_{q_i}$   | - | Force in $q_i$ -direction  |
| $\bar{F}$   | - | Dimensionless liquid force, $F/\rho g a^3$   |
| $F(s)$      | - | Laplace transform of the force $F$   |
| $F(t)$      | - | Arbitrary function of time   |
| $g$         | - | Acceleration due to gravity  |
| $\bar{g}_n$ | - | Damping factor of $n^{th}$ liquid mode   |
| $H_n$       | - | z-coordinate for point of attachment of $n^{th}$ pendulum  |
| $h$         | - | Depth of liquid measured from tank bottom  |
| $h_n$       | - | z-coordinate of $n^{th}$ slosh mass  |
| $I$         | - | Mass moment of inertia of fixed mass   |



NOTATION (continued)

|                 |   |   |
|-----------------|---|---|
| $\bar{I}$       | - | Effective mass moment of inertia of fluid                                       |
| $I_d$           | - | Mass moment of inertia of disc  |
| $I_{mn}$        | - | Mass moment of inertia of $mn^{th}$ slosh mass                                  |
| $I_s$           | - | Mass moment of inertia of solidified fluid                                      |
| $i$             | - | Imaginary unit  |
| $\hat{i}$       | - | Unit vector in x-direction  |
| $J$             | - | Mass moment of inertia of fixed mass  |
| $\bar{J}$       | - | Effective mass moment of inertia of fluid                                       |
| $J_{mn}$        | - | Mass moment of inertia of $mn^{th}$ slosh mass                                  |
| $J_\nu(\beta)$  | - | Bessel function of the 1 <sup>st</sup> kind of order $\nu$ and argument $\beta$ |
| $J'_\nu(\beta)$ | - | First derivative of $J_\nu(\beta)$ with respect to the argument $\beta$         |
| $\hat{j}$       | - | Unit vector in y-direction  |
| $K_{mn}$        | - | Torsional stiffness coefficient of shaft  |
| $k$             | - | Ratio of inner to outer radius of annular or annular-sector tank, $b/a$         |
| $\hat{k}$       | - | Unit vector in z-direction  |
| $L_n$           | - | Pendulum length   |
| $M$             | - | Fixed mass  |
| $M_n$           | - | Magnitude of moment after $n$ cycles  |
| $M_o$           | - | Magnitude of initial moment   |
| $M_{q_i}$       | - | Moment about $q_i$ -axis  |
| $M(s)$          | - | Laplace transform of the moment $M$   |
| $M_B(s)$        | - | Laplace transform of the moment $M$ on the baffle                               |
| $m$             | - | Mode of vibration in the $\phi$ -direction or y-direction                       |
| $m_L$           | - | Liquid mass   |



NOTATION (continued)

|             |   |  |
|-------------|---|--|
| $m_o$       | - | $n^{\text{th}}$ slosh mass   |
| $n$         | - | Normal direction<br>Mode of vibration in the r-direction or x-direction<br>Number of baffles |
| $p_{q_i}$   | - | Pressure in the $q_i$ -direction   |
| $p_o$       | - | Gauge pressure   |
| $q_i$       | - | Generalized coordinate   |
| $R$         | - | Radius of circular cylindrical tank  |
| $r$         | - | Cylindrical coordinate, $(r, \phi, z)$   |
| $r_1$       | - | Aspect ratio in xz-plane for rectangular tank, $h/a$   |
| $r_2$       | - | Aspect ratio in yz-plane for rectangular tank, $h/b$   |
| $r_3$       | - | Aspect ratio in xy-plane for rectangular tank, $a/b$   |
| $S, s$      | - | Laplace transform variable   |
| $T$         | - | Period of oscillation  |
| $t$         | - | Time   |
| $V$         | - | Potential function   |
| $V_m$       | - | Timewise maximum fluid velocity  |
| $\vec{u}$   | - | Rotation vector for vorticity, $\xi \hat{i} + \eta \hat{j} + \zeta \hat{k}$                  |
| $u$         | - | Velocity component in x-direction  |
| $\vec{V}$   | - | Velocity vector, $u \hat{i} + v \hat{j} + w \hat{k}$   |
| $\vec{V}_n$ | - | Velocity component normal to tank wall   |
| $\vec{V}_1$ | - | Velocity of container  |
| $v$         | - | Velocity component in y-direction  |
| $W, w$      | - | Baffle width measured normal to tank wall  |
| $w$         | - | Velocity component in z-direction  |
| $x$         | - | Rectangular coordinate, $(x, y, z)$<br>Displacement in x-direction                           |





# NOTATION (continued)

|               |   |  |
|---------------|---|--|
| $x_n$         | - | Translational displacement of $n^{\text{th}}$ slosh mass                               |
| $x_o, x_o$    | - | Amplitude of displacement function, $x = x_o e^{i\omega t}$ or $x = x_o \sin \omega t$ |
| $x(s)$        | - | Laplace transformed rectangular coordinate   |
| $Y_v(\beta)$  | - | Bessel function of the $2^{\text{nd}}$ kind of order $v$ and argument $\beta$          |
| $Y'_v(\beta)$ | - | First derivative of $Y_v(\beta)$ with respect to argument                              |
| $y$           | - | Rectangular coordinate, $(x,y,z)$<br>Translational displacement in y-direction         |
| $y_o$         | - | Amplitude of displacement function, $y = y_o e^{i\omega t}$                            |
| $z$           | - | Rectangular coordinate, $(x,y,z)$<br>Cylindrical coordinate, $(r,\phi,z)$              |

Note: The notation given in articles 5.1.5.2 and 5.1.7.3 is applicable to these articles only.



GREEK LETTERS

|                            |   |   |
|----------------------------|---|---|
| $\alpha$                   | - | Ratio of apex angle to $2\pi$ , $\tilde{\alpha}/2\pi$<br>Baffle width parameter in Miles' equation, $2a/W$                          |
| $\bar{\alpha}$             | - | Apex angle of sector tank or similar configuration<br>Effective baffle area for a baffle located below the undisturbed free surface |
| $\bar{\alpha}^*$           | - | Effective baffle area for a baffle located above the undisturbed free surface   |
| $\beta_{mn}$               | - | Angular displacement of $m^{th}$ sloss mass   |
| $\Gamma_n$                 | - | Generalized coordinate, $\lambda_n - \theta$  |
| $\epsilon_v, \xi_v$        | - | Roots of an eigenfunction where $v = mn, n$ or $(2m-1, n)$  |
| $\gamma, \gamma_s, \delta$ | - | Damping factor or damping ratio   |
| $\zeta, \zeta_w, \zeta_1$  | - | Surface displacement of liquid generally measured at tank wall  |
| $\eta_v$                   | - | Ratio of natural angular frequency to forcing frequency, $\omega_v/\omega$ , where $v = mn, n$ , or $(2m-1, n)$                     |
| $\theta$                   | - | Angular displacement of tank for pitching about y-axis  |
| $\theta_o$                 | - | Amplitude of displacement function, $\theta = \theta_o e^{i\omega t}$ or $\theta = \theta_o \sin \omega t$                          |
| $\theta(s)$                | - | Laplace transform of the displacement $\theta$  |
| $\lambda_n$                | - | Angular displacement of $n^{th}$ pendulum   |
| $\nu$                      | - | Viscosity of liquid   |
| $\rho$                     | - | Mass density of liquid  |
| $\phi$                     | - | Total velocity potential, $\phi_1 + \phi_2$   |
| $\phi_1$                   | - | Velocity potential of container   |
| $\phi_2$                   | - | Velocity potential of liquid  |
| $\phi$                     | - | Cylindrical coordinate, $(r, \phi, z)$<br>Angular displacement for roll about z-axis  |
| $\phi_o$                   | - | Amplitude of displacement function, $\phi = \phi_o e^{i\omega t}$ or $\phi = \phi_o \sin \omega t$                                  |



GREEK LETTERS (continued)

- $\chi$  - Angular displacement for pitching about x-axis
- $\chi_0$  - Amplitude of displacement function,  $\chi = \chi_0 e^{i\omega t}$
- $\psi$  - Angular displacement of disc relative to tank
- $\omega$  - Forcing frequency
- $\omega_\nu$  - Natural angular frequency for free liquid oscillations  
where  $\nu = mn, n$ , or  $(2m-1, n)$



# SPECIAL DEFINITIONS

$$\gamma_v = (\epsilon_v \text{ or } \ell_v) g/a\omega^2$$

$$\kappa_v = (\epsilon_v \text{ or } \ell_v) \frac{h}{a}$$

$$\rho_v = (\epsilon_v \text{ or } \ell_v) \frac{r}{a}$$

$$\sigma_v = (\epsilon_v \text{ or } \ell_v) k$$

$$\zeta_v = (\epsilon_v \text{ or } \ell_v) \frac{z}{a}$$

where  $v = mn, m, n, \text{ or } (2m-1, n)$

$$\bar{\phi}_m = \frac{m}{2\alpha} \phi,$$

$$\bar{\psi}_m = \frac{2m-1}{2\alpha} \psi$$





# DEFINITION OF PARAMETERS

## Sector Tank

$$\begin{pmatrix} \bar{A}_{mn} \\ \bar{C}_{mn} \end{pmatrix} = \frac{a \bar{b}_{mn}}{\epsilon_{mn} (\eta_{mn}^2 - 1) \cosh \kappa_{mn}} \begin{pmatrix} a_m \\ c_m \end{pmatrix} \left[ 2 \sinh \frac{\kappa_{mn}}{2} - \left( \gamma_{mn} + \frac{\kappa_{mn}}{2} \right) \cosh \frac{\kappa_{mn}}{2} \right]$$

$$\begin{pmatrix} \bar{B}_{mn} \\ \bar{D}_{mn} \end{pmatrix} = \frac{a \bar{b}_{mn}}{\epsilon_{mn} (\eta_{mn}^2 - 1) \cosh \kappa_{mn}} \begin{pmatrix} a_m \\ c_m \end{pmatrix} \left[ \left( \gamma_{mn} - \frac{\kappa_{mn}}{2} \right) \sinh \frac{\kappa_{mn}}{2} - 2 \cosh \frac{\kappa_{mn}}{2} \right]$$

$$a_m = \frac{\sin \bar{\alpha}}{\bar{\alpha}}, \quad m = 0$$

$$a_m = \frac{2 \bar{\alpha} (-1)^{m+1} \sin \bar{\alpha}}{m^2 \pi^2 - \bar{\alpha}^2}, \quad m = 1, 2, 3, \dots$$

$$\bar{b}_{mn} = \frac{2a \Gamma(m/4\alpha + 3/2)}{\epsilon_{mn} (1 - m^2/4\alpha^2 \epsilon_{mn}^2) \Gamma(m/4\alpha - 1/2) J_{m/2\alpha}^2(\epsilon_{mn})}$$

$$\sum_{\nu=0}^{\infty} \frac{(m/2\alpha + 2\nu + 1) \Gamma(m/4\alpha + \nu - 1/2)}{\Gamma(m/4\alpha + \nu + 5/2)} J_{m/2\alpha + 2\nu + 1}(\epsilon_{mn})$$

$$c_m = \frac{1 - \cos \bar{\alpha}}{\bar{\alpha}}, \quad m = 0$$

$$c_m = \frac{2 \bar{\alpha} (-1)^m \cos \bar{\alpha} - 1}{m^2 \pi^2 - \bar{\alpha}^2}, \quad m = 1, 2, 3, \dots$$



Sector Tank (Continued)

$$\bar{e}_{2m-1,n} = \frac{2 \left[ (2m-1)/4\alpha \right] \left[ (2m-1)/4\alpha + 1 \right] \left[ (2m-1)/4\alpha - 1 \right] \epsilon_{2m-1,n}}{\left\{ \epsilon_{2m-1,n}^2 - \left[ (2m-1)/2\alpha \right]^2 \right\} J_{(2m-1)/2\alpha}^2(\epsilon_{2m-1,n})} \\ + \sum_{\nu=0}^{\infty} \frac{\left[ (2m-1)/2\alpha + 2\nu + 1 \right] J_{(2m-1)/2\alpha + 2\nu + 1}(\epsilon_{2m-1,n})}{\left[ (2m-1)/2\alpha + \nu \right] \left[ (2m-1)/2\alpha + \nu + 1 \right] \left[ (2m-1)/2\alpha + \nu + 2 \right] \left[ (2m-1)/2\alpha + \nu - 1 \right]}$$

$$\bar{f}_{2m-1,n} = \frac{2 \left[ (2m-1)/2\alpha \right]}{\left\{ \epsilon_{2m-1,n}^2 - \left[ (2m-1)/2\alpha \right]^2 \right\} J_{(2m-1)/2\alpha}(\epsilon_{2m-1,n})}$$

$$L_0(\epsilon_{mn}) = \frac{2}{\epsilon_{mn}} \sum_{\nu=0}^{\infty} J_{m/2\alpha - 2\nu + 1}(\epsilon_{mn}) \quad \left( \operatorname{Re} \frac{m}{2\alpha} > 1 \right)$$

$$L_0(\epsilon_{2m-1,n}) = \frac{2}{\epsilon_{2m-1,n}} \sum_{\nu=0}^{\infty} J_{(2m-1)/2\alpha - 2\nu + 1}(\epsilon_{2m-1,n}) \quad \left[ \operatorname{Re} (2m-1)/2\alpha > 1 \right]$$

$$L_1(\epsilon_{2m-1,n}) = \frac{2m-1}{4\alpha \epsilon_{2m-1,n}} \sum_{\nu=0}^{\infty} \frac{\left[ (2m-1)/2\alpha + 2\nu + 1 \right]}{\left[ (2m-1)/4\alpha + \nu \right] \left[ (2m-1)/4\alpha + \nu + 1 \right]} \cdot$$

$$J_{(2m-1)/2\alpha + 2\nu + 1}(\epsilon_{2m-1,n})$$

$$L_2(\epsilon_{mn}) = \frac{\Gamma(m/2\alpha - 3/2)}{\epsilon_{mn} \Gamma(m/4\alpha - 1/2)} \sum_{\nu=0}^{\infty} \frac{(m/2\alpha + 2\nu + 1) \Gamma(m/2\alpha + \nu - 1/2)}{\Gamma(m/4\alpha + \nu + 5/2)} J_{m/2\alpha + 2\nu + 1}(\epsilon_{mn})$$

$$L_2(\epsilon_{2m-1,n}) = \frac{\Gamma \left[ (2m-1)/2\alpha + 3/2 \right]}{\epsilon_{2m-1,n} \Gamma \left[ (2m-1)/4\alpha - 1/2 \right]} \sum_{\nu=0}^{\infty} \frac{\left[ (2m-1)/2\alpha + 2\nu + 1 \right] \Gamma \left[ (2m-1)/4\alpha + \nu - 1/2 \right]}{\Gamma \left[ (2m-1)/4\alpha + \nu + 5/2 \right]} \cdot$$

$$J_{2m-1/2\alpha + 2\nu + 1}(\epsilon_{2m-1,n})$$



# Quarter-Sector Tank

$$\bar{A}_{mn} = \frac{a \bar{a}_m \bar{b}_{mn}}{\epsilon_{mn} (\eta_{mn}^2 - 1) \cosh \kappa_{mn}} \left[ 2 \sinh \frac{\kappa_{mn}}{2} - \left( \frac{\kappa_{mn}}{2} - \gamma_{mn} \right) \cosh \frac{\kappa_{mn}}{2} \right]$$

$$\bar{a}_m = \frac{2}{\pi} \cdot m \cdot 0$$

$$\bar{a}_m = \frac{(-1)^{m+1}}{\pi (m^2 - 1/4)}, \quad m = 1, 2, 3, \dots$$

$$\bar{B}_{mn} = \frac{a \bar{a}_m \bar{b}_{mn}}{\epsilon_{mn} (\eta_{mn}^2 - 1) \cosh \kappa_{mn}} \left[ \left( \gamma_{mn} - \frac{\kappa_{mn}}{2} \right) \sinh \frac{\kappa_{mn}}{2} - 2 \cosh \frac{\kappa_{mn}}{2} \right]$$

$$\bar{b}_{mn} = \frac{16a(m^2 - 1/4) \epsilon_{mn}}{(\epsilon_{mn}^2 - 4m^2) J_{2m}^2(\epsilon_{mn})} \sum_{\nu=0}^{\infty} \frac{J_{2m+2\nu-1}(\epsilon_{mn})}{(2m+2\nu+3)(2m+2\nu-1)}$$

$$e_n = \frac{4}{(\epsilon_n^2 - 4) J_2^2(\epsilon_n)}$$

$$e_{2m-1,n} = \frac{2m(2m-1) \epsilon_{2m-1,n}}{(m-1) \left[ \epsilon_{2m-1,n}^2 - 4(2m-1)^2 \right] J_{4m-2}^2(\epsilon_{2m-1,n})} \cdot \sum_{\nu=0}^{\infty} \frac{(4m+2\nu-1) J_{4m+2\nu-1}(\epsilon_{2m-1,n})}{(2m+\nu)(2m+\nu+1)(2m+2\nu-1)(2m+2\nu-2)}$$

$$f_n = \frac{-8}{(\epsilon_n^2 - 4) J_2^2(\epsilon_n)} \sum_{\nu=0}^{\infty} \frac{J_{2\nu+4}(\epsilon_n)}{(\nu+1)(\nu+3)}$$



Quarter-Sectoral Tank (Continued)

$$f_{2m-1,n} = \frac{4(2m-1)}{\left[ \epsilon_{2m-1,n}^2 - 4(2m-1)^2 \right]} J_{4m-2}(\epsilon_{2m-1,n})$$

$$L_o(\epsilon_n) = \frac{2}{\epsilon_n} \sum_{\nu=0}^{\infty} J_{3+2\nu}(\epsilon_n)$$

$$L_o(\epsilon_{mn}) = \frac{2}{\epsilon_{mn}} \sum_{\nu=0}^{\infty} J_{2m+2\nu+1}(\epsilon_{mn}) \quad (\text{Re } 2m > -1)$$

$$L_o(\epsilon_{2m-1,n}) = \frac{2}{\epsilon_{2m-1,n}} \sum_{\nu=0}^{\infty} J_{4m-2\nu-1}(\epsilon_{2m-1,n})$$

$$L_1(\epsilon_n) = \frac{1}{\epsilon_n} \sum_{\nu=0}^{\infty} \frac{2\nu+3}{(\nu+1)(\nu+2)} J_{2\nu+3}(\epsilon_n)$$

$$L_1(\epsilon_{2m-1,n}) = \frac{2m-1}{\epsilon_{2m-1,n}} \sum_{\nu=0}^{\infty} \frac{4m+2\nu-1}{(2m+\nu-1)(2m-\nu)} J_{4m+2\nu-1}(\epsilon_{2m-1,n})$$

$$L_2(\epsilon_n) = \frac{\Gamma(7/2)}{\epsilon_n \Gamma(3/2)} \sum_{\nu=0}^{\infty} \frac{(2\nu+3)\Gamma(\nu+1/2)}{\Gamma(\nu+7/2)} J_{2\nu+3}(\epsilon_n)$$

$$L_2(\epsilon_{mn}) = \frac{2(4m^2-1)}{\epsilon_{mn}} \sum_{\nu=0}^{\infty} \frac{J_{2m+2\nu-1}(\epsilon_{mn})}{(2m+2\nu+3)(2m-2\nu-1)}$$

$$L_2(\epsilon_{2m-1,n}) = \frac{2(4m-1)(4m-3)}{\epsilon_{2m-1,n}} \sum_{\nu=0}^{\infty} \frac{J_{4m+2\nu-1}(\epsilon_{2m-1,n})}{(4m+2\nu+1)(4m+2\nu-3)}$$





### Eighth-Sectored Tank

$$\bar{\epsilon}_{mn} = \frac{4m(4m^2-1) \epsilon_{mn}}{(\epsilon_{mn}^2 - 16m^2) J_{4m}^2(\epsilon_{mn})} \sum_{\mu=0}^{\infty} \frac{(4m+2\mu+1) J_{4m+2\mu+1}(\epsilon_{mn})}{(2m+\mu)(2m+\mu+1)(2m+\mu-1)(2m+\mu+2)}$$

$$\bar{f}_{mn} = \frac{8m}{(\epsilon_{mn}^2 - 16m^2) J_{4m}(\epsilon_{mn})}$$

$$L_2(\epsilon_{mn}) = \frac{2m}{\epsilon_{mn}} \sum_{\mu=0}^{\infty} \frac{(4m+2\mu+1) J_{4m+2\mu+1}(\epsilon_{mn})}{(2m+\mu)(2m+\mu+1)}$$

### Annular Tank

$$A_n = \frac{2[2/\pi \xi_n - k C_1(\sigma_n)]}{(4/\pi^2 \xi_n^2)(\xi_n^2 - 1) + C_1^2(\sigma_n)(1 - \sigma_n^2)}$$

### Annular-Sector Tank

$$\left( \frac{A}{C_{mn}} \right) = \frac{a b_{mn}}{\xi_{mn} (n_{mn}^2 - 1) \cosh \kappa_{mn}} \left( \frac{a_m}{c_m} \right) \left[ 2 \sinh \frac{\kappa_{mn}}{2} - \left( \frac{\kappa_{mn}}{2} + \gamma_{mn} \right) \cosh \frac{\kappa_{mn}}{2} \right]$$

$$\left( \frac{B}{D_{mn}} \right) = \frac{a b_{mn}}{\xi_{mn} (n_{mn}^2 - 1) \cosh \kappa_{mn}} \left( \frac{a_m}{c_m} \right) \left[ \left( \gamma_{mn} - \frac{\kappa_{mn}}{2} \right) \sinh \frac{\kappa_{mn}}{2} - 2 \cosh \frac{\kappa_{mn}}{2} \right]$$

$$a_m = \frac{\sin \bar{\alpha}}{\bar{\alpha}}, \quad m = 0$$

$$a_m = \frac{2\bar{\alpha}(-1)^{m+1} \sin \bar{\alpha}}{m^2 \pi^2 - \bar{\alpha}^2}, \quad m = 1, 2, 3, \dots$$



Annular-Sector Tank (continued)

$$b_{mn} = \frac{2a N_2(\xi_{mn})}{\left[4/\pi^2 \xi_{mn}^2 - k^2 C_{m/2\alpha}^2(\sigma_{mn})\right] - (m^2/4\alpha^2 \xi_{mn}^2) \left[4/\pi^2 \xi_{mn}^2 - C_{m/2\alpha}^2(\sigma_{mn})\right]}$$

$$c_m = \frac{1 - \cos \bar{\alpha}}{\bar{\alpha}}, \quad m = 0$$

$$c_m = \frac{2\bar{\alpha} [(-1)^m \cos \bar{\alpha} - 1]}{m^2 \pi^2 - \bar{\alpha}^2}, \quad m = 1, 2, 3, \dots$$

$$g_{2m-1,n} = \frac{\int_{\sigma_{2m-1,n}}^{\xi_{2m-1,n}} \rho_{2m-1,n}^3 C_{(2m-1)/2\alpha}(\rho_{2m-1,n}) d\rho_{2m-1,n}}{\xi_{2m-1,n}^2 \int_{\sigma_{2m-1,n}}^{\xi_{2m-1,n}} \rho_{2m-1,n} C_{(2m-1)/2\alpha}^2(\rho_{2m-1,n}) d\rho_{2m-1,n}}$$

$$l_{2m-1,n} = \frac{\int_{\sigma_{2m-1,n}}^{\xi_{2m-1,n}} \rho_{2m-1,n}^{1+(2m-1)/2\alpha} C_{(2m-1)/2\alpha}(\rho_{2m-1,n}) d\rho_{2m-1,n}}{\xi_{2m-1,n}^{(2m-1)/2\alpha} \int_{\sigma_{2m-1,n}}^{\xi_{2m-1,n}} \rho_{2m-1,n} C_{(2m-1)/2\alpha}^2(\rho_{2m-1,n}) d\rho_{2m-1,n}}$$

$$q_{2m-1,n} = \frac{\xi_{2m-1,n}^{(2m-1)/2\alpha} \int_{\sigma_{2m-1,n}}^{\xi_{2m-1,n}} \rho_{2m-1,n}^{1-(2m-1)/2\alpha} C_{(2m-1)/2\alpha}(\rho_{2m-1,n}) d\rho_{2m-1,n}}{\int_{\sigma_{2m-1,n}}^{\xi_{2m-1,n}} \rho_{2m-1,n} C_{(2m-1)/2\alpha}^2(\rho_{2m-1,n}) d\rho_{2m-1,n}}$$



Annular-Sector Tank (continued)

$$N_j(\xi_{mn}) = Y'_{m/2\alpha}(\xi_{mn}) \frac{\Gamma(m/4\alpha+1/2+j/2)}{\xi_{mn} \Gamma(m/4\alpha+1/2-j/2)} \sum_{\mu=0}^{\infty} \frac{(m/2\alpha+2\mu+1)\Gamma(m/4\alpha+\mu+1/2-j/2)}{\Gamma(m/4\alpha+\mu+3/2+j/2)}.$$

$$\begin{aligned} & \left[ J_{m/2\alpha+2\mu-1}(\xi_{mn}) - k^j J_{m/2\alpha+2\mu+1}(k\xi_{mn}) \right] \\ & + J'_{m/2\alpha}(\xi_{mn}) \left\{ \frac{1}{\pi} \sum_{\mu=0}^{\left(\frac{m}{2}-1\right)} \frac{\xi_{mn}^{2\mu-m/2\alpha} (m/2\alpha-\mu-1)! (1-k^{2\mu-m/2\alpha+1+j})}{(2\mu-m/2\alpha+1+j) 2^{2\mu-m/2\alpha}} \right. \\ & + \frac{2}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu (1-k^{m/2\alpha+2\mu-1+j}) \xi_{mn}^{m/2\alpha+2\mu}}{\mu! (m/2\alpha+\mu)! (m/2\alpha+2\mu+1+j) 2^{m/2\alpha+\mu}} \\ & \left. + \frac{2k^{j+1}}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu (k\xi_{mn})^{m/2\alpha+2\mu}}{\mu! (m/2\alpha+\mu)! (m/2\alpha+2\mu+1+j) 2^{m/2\alpha+2\mu}} \right\}. \end{aligned}$$

$$\begin{aligned} & \left[ \ln \left( \frac{k\xi_{mn}}{2} \right) - \frac{1}{2} \psi(\mu+1) - \frac{1}{2} \psi \left( \frac{m}{2\alpha} + \mu + 1 \right) \right] \\ & - \frac{2}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu \xi_{mn}^{m/2\alpha+2\mu}}{\mu! (m/2\alpha+\mu)! (m/2\alpha+2\mu+1+j) 2^{m/2\alpha+2\mu}}. \end{aligned}$$

$$\left[ \ln \left( \frac{\xi_{mn}}{2} \right) - \frac{1}{2} \psi(\mu+1) - \frac{1}{2} \psi(m/2\alpha+\mu+1) \right] \Bigg\}, \quad j = 0, 1, 2, \dots$$

where  $\psi(z) = \frac{d \ln \Gamma(z)}{dz} = -\gamma + (z-1) \sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)(\lambda+z)}$

$z$  is the argument,  $(\mu+1)$  and  $(m/2\alpha+\mu+1)$  is the above equation.

$\gamma = 0.5772157$  is Euler's constant.

$N_j(\xi_{2m-1,n}) = N_j(\xi_{mn})$  if in the above equation for  $N_j(\xi_{mn})$ ,  $m$  is replaced by  $(2m-1)$ .



## I. Introduction

Fuel sloshing is defined as any periodic motion of a contained liquid propellant and results from the missile's oscillatory motion about its flight trajectory. The most likely causes of such disturbances are gust loads, control modes, and structural modes. If any of these excitations have frequencies in the vicinity of the resonant frequency of a contained liquid, violent sloshing will occur. Since fuel and oxidizer amount to an extremely high percentage of the gross vehicle weight, the magnitudes of the liquid force and moment resultants are significant and cannot be neglected. A thorough knowledge of the magnitude and location of the external forces and moments acting on a space vehicle is required for making stability and structural investigations. Consequently, the effects of liquid propellants on their containers while undergoing forced vibrations have been investigated both analytically and experimentally. The results of such studies are found in Chapter IV.

The stability of a missile can be increased by reducing the force and moment resultants caused by the oscillating propellant and increasing the sloshing frequency. Increasing the eigen frequencies of the liquid decreases the depth below the surface to which the disturbances extend. The above fact can be accomplished by utilizing longitudinal partitions which effectively reduce the sloshing liquid mass and increase the frequencies of liquid oscillations. Also various types of baffles and floats can be used to help damp liquid motion. To evaluate the effectiveness of such anti-slosh devices, its damping factor and effect on the eigen frequencies of the fluid need to be known. Consequently, there has been a considerable amount of analytical and experimental work done in this area. The results of such investigations are found in Chapter V.

The exact formulation and solution to the problem of oscillations of a contained liquid with a free surface are extremely complex. To simplify the matter, it is convenient to assume that the fluid is nonviscous and the flow is irrotational, thereby permitting the use of potential theory. The problem then reduces to obtaining the velocity potential from Laplace's equation solved with the appropriate boundary conditions.

The overall problem of missile stability can be investigated more easily if the liquid is replaced with a dynamically equivalent mechanical model. The equations of motion for a model are derivable from either Lagrange's equations or simple equilibrium theory and from these equations, the force and moment resultants can be obtained. The characteristics of the model are then determined so that the force and moment resultants, frequencies of oscillation, and mass and inertial characteristics are identically equal to those of the liquid. Also by using equivalent mechanical models, the nonlinear nature of forced damped liquid oscillations can be approximated by equivalent linear damping. This makes it possible to obtain finite results near the resonant frequencies of the liquid, which was not possible in the fluid analysis, since the liquid was assumed to be inviscid.



## II. Fluid Theory (9)

2.1 Assumptions

The difficulty in formulating and obtaining the exact solution to the problem of oscillations of a contained liquid is simplified by making several assumptions concerning the nature of the fluid and the type of flow encountered. These assumptions are: (1) the fluid is both incompressible ( $\rho = \text{constant}$ ) and frictionless ( $\mu = 0$ ), (2) the flow is irrotational ( $\vec{\omega} = 0$ ), and (3) there are no sources or sinks; that is, the container is neither draining nor filling.

2.2 Basic Equations

Irrotationality is a necessary and sufficient condition that the velocity  $\vec{V}$  can be expressed as the gradient of a velocity potential  $\phi$ , that is,

$$\nabla\phi = \vec{V} \quad (2-1)$$

Because of incompressibility and the nonexistence of sources and sinks, the equation of continuity reduces to Laplace's equation,

$$\nabla^2\phi = 0 \quad (2-2)$$

For an incompressible and frictionless fluid, irrotational flow, and conservative body forces, those forces derivable from a potential function, Bernoulli's energy equation is

$$\frac{p}{\rho} + \frac{\vec{V}^2}{2} + U + \frac{\partial\phi}{\partial t} = F(t)$$

## II. Fluid Theory (9) (continued)

where  $U$  is a potential function. Since gravity is the only important body force, then  $U = gz$ , and since the velocity is assumed to be small, the term containing  $\vec{v}^2$  is neglected. Also since any arbitrary function of time can be added to the velocity potential without changing the flow it represents, the function  $F(t)$  can be conveniently absorbed by  $\phi$ . Consequently, the equation of motion reduces to

$$\frac{1}{\rho} \frac{\partial p}{\partial t} + gz + \frac{\partial \phi}{\partial t} = 0 \quad (2-3)$$

From this expression, the pressure  $p$  is found to be

$$p = -\rho \left[ \frac{\partial \phi}{\partial t} + gz \right] \quad (2-4)$$

For zero or uniform surface gauge pressure ( $p = 0$  or  $p = p_0 = \text{const.}$ ) the surface displacements can be found as follows:

$$z_{\text{surface}} = z = \zeta = -\frac{1}{g} \frac{\partial \phi}{\partial t} \quad (2-5)$$

where, in the case of uniform surface gauge pressure, the constant  $\frac{p_0}{\rho}$  is absorbed by  $\phi$ . The resultant liquid force acting on a container is then given by

$$F_{q_i} = \int_A p_{q_i} dA \quad (2-6)$$

where  $q_i$  is a generalized coordinate and  $F_{q_i}$  and  $p_{q_i}$  are components of the force and pressure distributions in the  $q_i$ -direction. Taking the  $z$ -axis to be the longitudinal axis of the container, the resultant liquid moment about a line perpendicular to the  $xz$ -plane and passing through the point  $(0,0,\bar{z})$  is given by

$$M_{z=\bar{z}} = \int_{A_{\text{SIDE}}} p_{q_i} (z + \bar{z}) dA + \int_{A_{\text{BOTTOM}}} p_{q_i} x dA \quad (2-7)$$

## II. Fluid Theory (9) (continued)

2.3 Boundary Conditions

2.3.1 Velocity Potential. The total velocity potential  $\phi$  is composed of two parts:  $\phi_1$ , the velocity potential of the container which is assumed to be small and  $\phi_2$ , the velocity potential of the liquid. These potentials must satisfy Laplace's equation in the following manner:

$$\nabla^2 \phi = \nabla^2 (\phi_1 + \phi_2) = 0 \quad (2-8)$$

Since for a particular excitation the container velocity is known, the velocity potential of the container can easily be determined from equation (2-1)

$$\nabla \phi_1 = \vec{V}_1 \quad (2-9)$$

where the resulting constant of integration can be absorbed by  $\phi_2$ .

2.3.2 Free Oscillations. Consideration of free oscillations is necessary in order to determine the natural frequencies of liquids in variously shaped containers. For the occurrence of free liquid oscillations, the container must be at rest ( $\vec{V}_1 = 0$ ), thus  $\phi_1 = \text{constant}$ . Consequently,  $\phi = \phi_2$  since the constant can be absorbed by  $\phi_2$ . In order to avoid separation over the wetted surface, the velocity of the fluid normal to the container wall must be identical to that of the container itself, that is,

$$\frac{\partial \phi_2}{\partial n} = \vec{V}_n = 0 \quad (2-10)$$

Also, the pressure variation over the free surface must be equal to zero. Accordingly, the linearized free surface condition is

$$\frac{\partial^2 \phi_2}{\partial t^2} + g \frac{\partial \phi_2}{\partial z} = 0 \quad (2-11)$$

## II. Fluid Theory (9) (continued)

2.3.2 Forced Oscillations. When the container is subjected to harmonic excitation, the total velocity potential,  $\phi = \phi_1 + \phi_2$ , must be considered. Since over the wetted surface the normal velocity of the fluid must be the same as that of the container, it follows that

$$\frac{\partial \phi}{\partial n} = \vec{V}_n \quad (2-12)$$

The linearized free surface condition corresponding to zero pressure variation at the free surface is

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \quad (2-13)$$

The free surface boundary condition is actually composed of both a zero-pressure variation condition and a kinematic condition. In the kinematic condition, second and higher order terms were neglected thereby linearizing the free surface boundary condition. When the amplitude of the fluid is large relative to the tank dimensions, the effect of nonlinearity can become important, i.e., the magnitude of the higher order effects might not be negligible. Therefore, in order that the slope of the free surface be small, the resulting frequency of fluid oscillation must not be in the immediate neighborhood of a natural free surface frequency. Even so, relatively large amplitudes are possible when the container is also large. Also, it should be noted that the free surface condition given here and in Article 2.3.2 above is valid only in the case of a constant gravitational field. Consequently, the information presented in this handbook is applicable only for ground test applications.

The expression for the natural frequencies for free liquid oscillations is as follows:

## II. Fluid Theory (9) (continued)

$$\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \left( \frac{h}{a} \epsilon_{mn} \right) \quad (2-14)$$

where

$g$  = the acceleration due to gravity

$a$  = the radius of the tank

$h$  = the depth of liquid in the tank

$\epsilon_{mn}$  = the roots of an eigen function

It is observed that the eigen frequencies are proportional to the square root of  $g$ . However, when the vehicle experiences a large acceleration normal to the mean free surface of the fluid,  $g$  becomes the effective longitudinal acceleration of the missile. Therefore, the natural frequencies of the liquid remain a constant while the vehicle is at rest but vary when the missile is in flight. It is also seen that the frequencies are inversely proportional to the square root of  $a$  and thereby decrease with an increase in the cross-sectional area of the container. The influence of tank geometry upon the frequencies of the liquid is also exhibited by the value of  $\epsilon_{mn}$ . The liquid height has relatively no effect on the frequencies for  $h > a$ . As  $h$  decreases below  $a$ , the frequencies tend to zero.

#### 2.4 Initial Conditions

Assume the harmonic forcing function is equal to zero for all time  $t$  equal to and less than zero ( $t \leq 0$ ).



### III. Equivalent Mechanical Model Theory

An equivalent mechanical model is an assemblage of springs, dashpots, masses, and massless rods arranged in such a manner as to represent the dynamic behavior of a complex mechanical or non-mechanical system. Consideration is given here to those models which are dynamically equivalent to a liquid oscillating within its container. Dynamic equivalence is taken to mean the equivalence of force and moment resultants, frequencies of oscillation, and mass and inertial characteristics.

There are two reasons for using equivalent mechanical models to represent the sloshing behavior of a contained liquid. Since the problem of forced damped liquid oscillations is actually nonlinear in nature, an exact solution is practically impossible. However, through the use of an equivalent mechanical model, a good approximation can be made by introducing equivalent linear damping in the form of dashpots, thereby making it possible to obtain finite results near the resonant frequencies of the liquid. This was not possible in the fluid analysis, since the liquid was assumed to be inviscid. Secondly, the overall problem of missile stability is simplified in that the vehicle's equations of motion are not so complex.

When a container partially filled with fluid is excited, the liquid in the bottom of the tank is little disturbed, whereas the liquid near the free surface oscillates. Analogously, the model is composed of a fixed mass in the bottom of the container to represent the essentially rigid or non-sloshing part of the liquid and a movable mass near the top of the container to represent the sloshing part of the liquid. The sum of the fixed and movable masses of the model are taken equal to the total mass of the liquid and the ratio of the movable mass to the total mass of the liquid increases

as the depth of the liquid in the container decreases. Also, the fixed mass and the movable mass (when the fluid is not oscillating) are located along the longitudinal axis of the container at such distances from the tank bottom as to yield the same inertial characteristics for rotation about any axis perpendicular to and passing through the longitudinal axis. Therefore, the fixed mass is located near the center of gravity of the undisturbed fluid and the location of the movable mass shifts towards the center of gravity of the fluid as the liquid level decreases.

The kind of movable mass to be used and its restraints are dependent on the type of excitation encountered, namely: harmonic translation, harmonic pitching, or harmonic roll. For a particular model, harmonic translation and harmonic pitching have the following relationship: (a) translation refers to an excitation in a direction perpendicular to the longitudinal axis of the container while (b) pitching implies an excitation about an axis perpendicular to the direction of translation and the container's longitudinal axis. For translational and pitching excitations, two possibilities exist to represent a movable mass and its constraints; either a spring-mass system whose motion is restrained by springs and dashpots aligned in the direction of translation or a pendulum-mass system having its massless lever arm attached to the longitudinal axis of the container. Because "small oscillation theory" has been assumed, the amplitude of the pendulum motion must be small, thereby maintaining the pendulum mass at a relatively constant distance from the container bottom. Therefore the inertial characteristics of the pendulum mass are essentially the same as those of the spring mass.

For roll excitation, which is always taken to be about the longitudinal



axis of the container, two possibilities exist for representing a movable mass and its constraints; (a) a torsional pendulum whose shaft coincides with the container's longitudinal axis or (b) a torsional spring-mass system attached to the longitudinal axis of the container.

Because of the defined arrangement of the constraints of movable masses with respect to the direction of excitation, the model becomes a single degree of freedom system, i.e., only one generalized coordinate is necessary to specify the position of the movable mass at any particular time. Note, however, that when a container partially filled with fluid is excited, the liquid can oscillate in various modes of vibration depending on the frequency and amplitude of the forcing function. Thus, for a circular cylindrical tank, or some other similar configuration, it is possible to have modes of vibration in both the radial and tangential direction. In other words, an oscillating liquid can exhibit several degrees of freedom. In order for a model to simulate these various modes of vibration, it is necessary to add additional movable masses to the model. Each new mass increases the model's degree of freedom by one and thereby makes it possible to represent an additional liquid mode of oscillation. For this reason the subscript  $mn$  is used on various parameters in order to clarify which mode of vibration is of interest.  $m$  is to represent the  $m$ th mode of vibration in the tangential direction and  $n$  is to represent the  $n$ th mode in the radial direction for circular cylindrical tanks or similar configurations. In many cases, the only  $m$ th mode to be excited is  $m=1$ . Thus, when only the subscript  $n$  is used, it is to be understood that  $m=1$ .

In order for a particular movable mass to oscillate at the same

frequency as the liquid mode it represents, the following definitions have to be made: (1) the spring constant is defined as the slosh mass times the square of the natural angular frequency of the fluid, (2) the length of the pendulum axis is defined as the acceleration due to gravity over the square of the natural angular frequency of the fluid, and (3) the damping coefficient is defined as the slosh mass times the damping factor times the natural angular frequency of the fluid. Also, for the torsional pendulum, the torsional stiffness of the shaft is defined as the mass moment of inertia of the disc times the natural angular frequency and the damping coefficient of the disc is defined as the mass moment of inertia times the damping factor times the square of the natural angular frequency. The damping factor must be determined either analytically or experimentally (see Chapter V).

Component values of the model are determined so that the forces and moments exerted by the model are identically equal to those exerted by the liquid. This is done by making a term-wise comparison of the force and moment equations derived from hydrodynamic aspects with those derived from the model. In order to put the fluid and model equations into similar form for comparison, it is many times necessary to expand certain terms of these equations into series. Even though the model equations contain damping terms and the fluid equations do not, the comparison of equations is still valid because the modes of vibration change little as long as the amount of damping remains small.

## IV. Rigid Tanks

## 4.1 Introduction

The following sections present results of analytical studies of liquid oscillations in containers of various shapes when subjected to different types of excitation. Each section is concerned with a particular container shape, either circular cylindrical, sector, quarter-sectored, eighth-sectored, annular, annular-sector, or rectangular and is preceded by a table of contents, a diagram showing the container and its coordinate system, and other information pertinent to the material that follows. The containers are assumed to have exhibited rigid-body behavior. The boundary conditions and resulting fluid velocity potentials, natural frequencies and liquid force and moment resultants are given along with equivalent mechanical models. The elements of these models are defined both analytically and graphically.

The orthogonal coordinate systems to which the force and moment resultants are referenced are identical to those shown in the figures, the coordinate origins always being located at the center of gravity of the undisturbed fluid. However, in the fluid analysis, it is frequently convenient to translate the coordinate origin along the  $z$ -axis to the free surface of the undisturbed liquid, thereby referencing the velocity potential to such a system. This fact presents no difficulty since equation (2-7), used for calculating the resultant moment, gives the moment referenced to any point  $(0,0,\bar{z})$ . Therefore, it is necessary to take  $\bar{z} = -h/2$  when the velocity potential is referenced to the fluid surface and  $\bar{z} = 0$  when the velocity potential is referenced to the center of gravity of the undisturbed fluid.

The orthogonal coordinate system to which a particular velocity

potential is referenced is determined in the following manner. The  $z$ -axis is always identical to that shown in the appropriate figure. The location of the coordinate origin along the  $z$ -axis will be either at the center of gravity of the undisturbed fluid or at the undisturbed free liquid surface. Upon examination of the  $z$ -coordinate at which the corresponding free surface boundary condition is evaluated, the location of the coordinate origin is obvious.

Also, it should be mentioned that, for a particular container, superposition of force and moment resultants resulting from several types of excitations is possible because of the linearized theory used herein.

## IV. Rigid Tanks

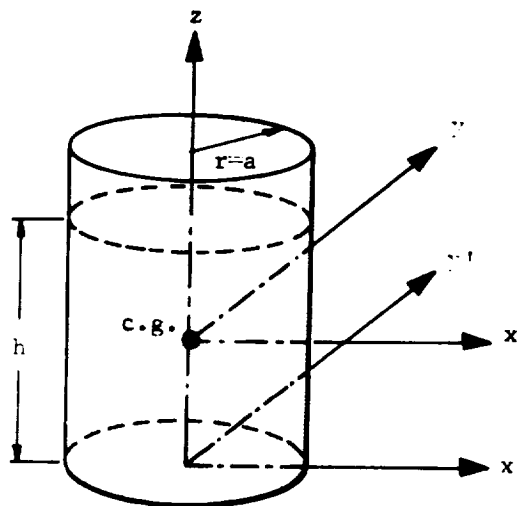
## 4.2 Circular Cylindrical Tank

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## IV. Rigid Tanks

## 4.2 Circular Cylindrical Tank



Container: The tank is a right circular cylinder of radius  $a$  and is filled with a liquid to a depth  $h$ .

Coordinate System: The origin of the  $xyz$ -system is located at the center of gravity of the undisturbed fluid and the origin of the  $x'y'z'$  system is located at the geometric center of the tank base.

References: (4,44,and 52)

Comments: The terms  $\bar{I}$  and  $\bar{c}$ , occurring in the model elements, must be experimentally determined, generally from torsion spring experiments of a sealed container of fluid. Their values are dependent upon container shape and the type of liquid.

Table 4-1. Boundary Conditions, Velocity Potential, and Natural Frequency

| Circular Cylindrical Tank   |  | Excitation: Harmonic Translation   |
|---|--|--|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .   |  |  |
| 1. Boundary conditions:   |  |  |
| (a) $\left(\frac{\partial \phi}{\partial r}\right)_{r=a} = i\omega x_0 e^{i\omega t} \cos \phi$   | (b) $\left(\frac{\partial \phi}{\partial z}\right)_{z=-h} = 0$ | (c) $\left(\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z}\right)_{z=0} = 0$ |
| 2. Velocity Potential:  |  |  |
| $\phi = i\omega x_0 e^{i\omega t} \cos \phi \left[ \frac{r}{a} + \sum_{n=0}^{\infty} \frac{2 J_1(\rho_n) \cosh(\kappa_n + \zeta_n)}{(\epsilon_n^2 - 1)(\eta_n^2 - 1) J_1(\epsilon_n) \cosh \kappa_n} \right]$ |  |  |
| 3. Natural angular frequency:   |  |  |
| $\omega_n^2 = \frac{g}{a} \epsilon_n \tanh \kappa_n$  |  |  |
| where $\epsilon_n$ are roots of $J_1'(\epsilon_n) = 0$ . (see Table A-3)  |  |  |

Table 4-1. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

| Circular Cylindrical Tank   |   | Excitation: Harmonic Pitching   |
|---|---|---|
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ .  |   |   |
| 1. Boundary conditions:   |   |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = -i\omega z \theta_0 e^{i\omega t} \cos \phi$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = i\omega r \theta_0 e^{i\omega t} \cos \phi$ | (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = 0$ |
| 2. Velocity potential:  |   |   |
| $\phi = -i\omega a^2 \theta_0 e^{i\omega t} \cos \phi \left\{ \frac{r}{2} z + \sum_{n=0}^{\infty} \frac{2 J_1(\rho_n)}{\epsilon_n (\epsilon_n^2 - 1)(\eta_n^2 - 1) J_1(\epsilon_n) \cosh \kappa_n} \right.$ $\left. \left[ \left( \gamma_n + \frac{\kappa_n}{2} \right) \cosh \left( \frac{\kappa_n}{2} + \zeta_n \right) - 4 \gamma_n \sinh \frac{\kappa_n}{2} \sinh \zeta_n - 2 \sinh \left( \frac{\kappa_n}{2} - \zeta_n \right) \right] \right\}$ |   |   |
| 3. Natural angular frequency:   |   |   |
| $\omega_n^2 = \frac{g}{a} \epsilon_n \tanh \kappa_n$  |   |   |
| where $\epsilon_n$ are roots of $J_1'(\epsilon_n) = 0$ . (see Table A-3)  |   |   |



Table 4-2. Liquid Force and Moment Resultants

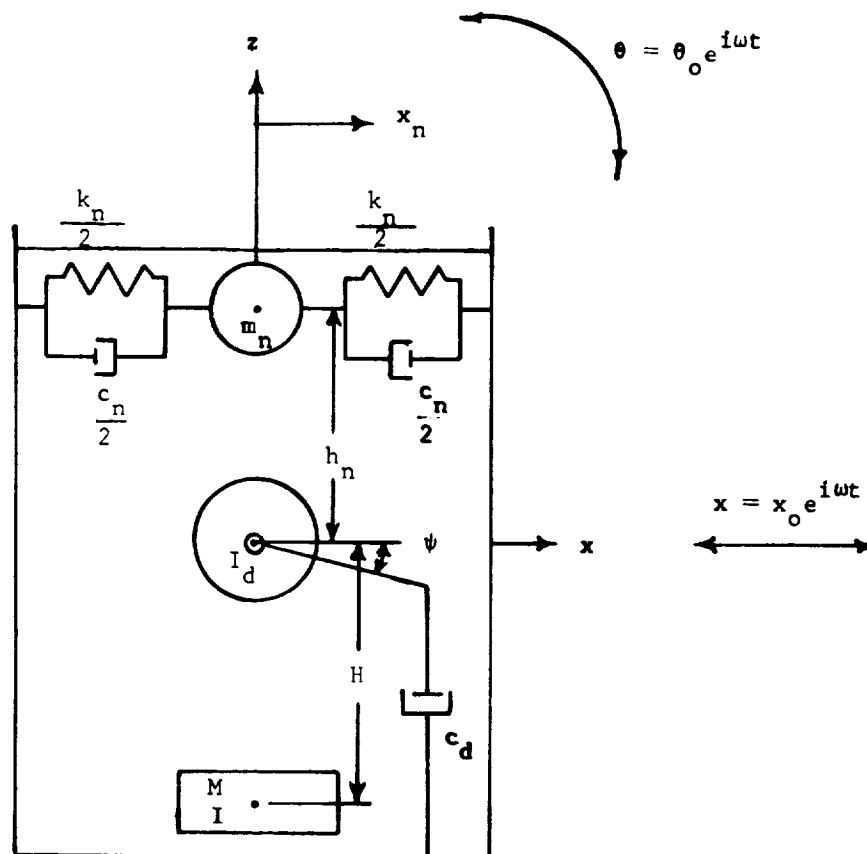
| Circular Cylindrical Tank  | Excitation: Harmonic Translation or Pitching |
|--|--|
| Translation in the x-direction, $x_0 e^{i\omega t}$ .  |  |
| 1. $F_x = m_L^2 \omega^2 x_0 e^{i\omega t} \left[ 1 + 2 \sum_{n=0}^{\infty} \frac{\tanh \kappa_n}{\kappa_n (\epsilon_n^2 - 1)(\eta_n^2 - 1)} \right]$  |  |
| 2. $M_y = m_L a \omega^2 x_0 e^{i\omega t} \left[ \frac{1}{4(h/a)} + 2 \sum_{n=0}^{\infty} \frac{(1/\kappa_n)(2/\cosh \kappa_n - 1) + (1/2) \tanh \kappa_n}{\epsilon_n (\epsilon_n^2 - 1)(\eta_n^2 - 1)} \right]$  |  |
| Pitching about the y-axis, $\theta_0 e^{i\omega t}$ .  |  |
| 1. $F_x = -m_L g \theta_0 e^{i\omega t} - 2 m_L a \omega^2 \theta_0 e^{i\omega t} \sum_{n=0}^{\infty} \frac{(2/\kappa_n)(1/\cosh \kappa_n - 1) + (\gamma_n/\kappa_n + 1/2) \tanh \kappa_n}{\epsilon_n (\epsilon_n^2 - 1)(\eta_n^2 - 1)}$   |  |
| 2. $M_y = -\frac{m_L g a \theta_0 e^{i\omega t}}{4(h/a)} - m_L^2 a^2 \omega^2 \theta_0 e^{i\omega t} \left\{ \frac{1}{12} \left(\frac{h}{a}\right)^2 - \frac{1}{8} + 2 \sum_{n=0}^{\infty} \frac{1}{\epsilon_n (\epsilon_n^2 - 1)(\eta_n^2 - 1)} \right. \\ \left. \left[ \left(1 - \frac{2\gamma_n}{\kappa_n}\right) \frac{2}{\epsilon_n \cosh \kappa_n} + \frac{1}{\epsilon_n} \left(\frac{\gamma_n}{\kappa_n} + \frac{1}{2}\right) + \frac{1}{\epsilon_n} \left(\frac{\kappa_n}{4} - \frac{3\gamma_n}{2} - \frac{4}{\kappa_n}\right) \tanh \kappa_n \right] \right\}$ |  |

Table 4-3. Model Analysis

| Circular Cylindrical Tank   | Spring-Mass Model |
|---|-------------------|
| Excitation: Harmonic Translation and/or Pitching  |                   |
| <p>Figure 4-1 shows a diagram of the spring-mass model used in representing the dynamic response of a liquid in a circular cylindrical tank when subjected to harmonic translation in the x-direction and/or pitching about the y-axis.</p> <p><u>Coordinate System:</u></p> <p>The origin is located at the center of gravity of the undisturbed liquid.</p> <p><u>Model Description:</u></p> <p>The components of the system are as follows:</p> <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the <math>z</math>-axis at a distance <math>H</math> below the coordinate origin.</li> <li>2. A set of movable masses <math>m_n</math> is distributed along the <math>z</math>-axis when the tank is at rest at distances <math>h_n</math> above the origin. These modal masses are constrained by spring-dashpot systems, having spring stiffness coefficients <math>k_n</math> and viscous damping coefficients <math>c_n</math>, to remain in the <math>xy</math>-plane and to move only in a direction parallel to the <math>x</math>-axis. Translational displacements of these masses with respect to the container are denoted by <math>x_n</math>.</li> <li>3. A massless disc having a moment of inertia <math>I_d</math> is located at the coordinate origin. Its motion is confined to rotation about the <math>y</math>-axis and is cushioned by a dashpot having a viscous damping coefficient <math>c_d</math>. The angular displacement of the disc relative to the tank is defined by <math>\psi</math>.</li> </ol> <p><u>Equations of Motion:</u></p> <p>The equations, obtained through Lagrange's equations, are as follows:</p> <ol style="list-style-type: none"> <li>1. Force Equation:</li> </ol> $F_x = -M(\ddot{x} + H\ddot{\theta}) - \sum_{n=1}^{\infty} m_n(\ddot{x} + \ddot{x}_n + h_n\ddot{\theta})$ <ol style="list-style-type: none"> <li>2. Moment Equation:</li> </ol> $M_y = -(I + MH^2)\ddot{\theta} - I_d(\ddot{\theta} + \ddot{\psi}) + g \sum_{n=1}^{\infty} m_n x_n - \sum_{n=1}^{\infty} m_n h_n(\ddot{x}_n + h_n\ddot{\theta})$ |                   |

Table 4.3 Model Analysis (continued)

| Circular Cylindrical Tank  | Spring-Mass Model |
|--|-------------------|
| Excitation: Harmonic Translation and/or Pitching   |                   |
| <p data-bbox="349 415 820 457"><u>Equations of Motion (continued):</u></p> <p data-bbox="422 464 673 495">3. Disc Equation:</p> $I_d (\ddot{\theta} + \ddot{\psi}) + c_d \dot{\psi} = 0$ <p data-bbox="422 579 771 611">4. Slosh-Mass Equation:</p> $m_n (\ddot{x} + \ddot{x}_n + h_n \ddot{\theta}) + m_n \bar{g}_n \omega_n \dot{x}_n + k_n x_n - m_n g \theta = 0$ <p data-bbox="349 699 1497 779">From these equations, the model force in the x-direction and the moment about the y-axis can be found (see Table 4-4).</p> |                   |



Circular Cylindrical Tank

Spring-Mass Model

Excitation: Harmonic Translation and/or Pitching

Figure 4-1. Equivalent Mechanical Model

Table 4-4. Model Force and Moment Resultants

| Circular Cylindrical Tank  | Spring-Mass Model |
|--|-------------------|
| Excitation: Harmonic Translation and/or Pitching   |                   |
| Translation in the x-direction, $x = x_o e^{i\omega t}$ .  |                   |
| 1. $F_x = m_L \omega^2 x_o e^{i\omega t} \left[ 1 + \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) \right]$  |                   |
| 2. $M_y = m_L h \omega^2 x_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h}{h} + \frac{g}{h\omega^2} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$  |                   |
| Pitching about the y-axis, $\theta = \theta_o e^{i\omega t}$ .   |                   |
| 1. $F_x = -m_L h \omega^2 \theta_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h}{h} + \frac{g}{h\omega^2} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$  |                   |
| 2. $M_y = -m_L h^2 \omega^2 \theta_o e^{i\omega t} \left[ \frac{1}{m_L h^2} + \frac{Mh^2}{m_L h^2} + \frac{I_d}{m_L h^2} \left( 1 - \frac{\omega^2 I_d^2}{c_d^2 + \omega^2 I_d^2} \right) + \sum_{n=0}^{\infty} \frac{m_n h^2}{m_L h^2} \right. \\ \left. + \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h}{h} + \frac{g}{h\omega^2} \right)^2 \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) \right] + i\omega \theta_o e^{i\omega t} \left( \frac{c_d \omega^2 I_d^2}{c_d^2 + \omega^2 I_d^2} \right)$ |                   |

Table 4-5. Model Elements

| Circular Cylindrical Tank                        |   | Spring-Mass Model |  |
|--|---|-------------------|--|
| Excitation: Harmonic Translation and/or Pitching |   |                   |  |
| Natural Frequency                                | $\omega_n^2 = \frac{g}{a} \epsilon_n \tanh \kappa_n$  | (fig. 4-2, 4-3)   |  |
| Spring Constant                                  | $k_n = m_n \omega_n^2$  | (fig. 4-4, 4-5)   |  |
| Damping Coefficient of Slosh Mass                | $c_n = m_n \bar{g}_n \omega_n$  |                   |  |
| Damping Coefficient of Disc                      | $c_d = \bar{c} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   | (fig. 4-6, 4-7)   |  |
| Ratio of Slosh Mass to Fluid Mass                | $\frac{m_n}{m_L} = \frac{2 \tanh (\kappa_n)}{(\epsilon_n^2 - 1) \kappa_n}$  | (fig. 4-8)        |  |
| Ratio of Fixed Mass to Fluid Mass                | $\frac{M}{m_L} = 1 - \sum_{n=0}^{\infty} \frac{m_n}{m_L}$   | (fig. 4-9, 4-10)  |  |
| Ratio of Slosh Mass Coordinate to Fluid Depth    | $\left  \frac{h_n}{h} \right  = \frac{1}{2} \left[ 1 - \frac{4}{\kappa_n} \tanh \frac{\kappa_n}{2} \right]$   | (fig. 4-11)       |  |
| Ratio of Fixed Mass Coordinate to Fluid Depth    | $\left  \frac{H}{h} \right  = \frac{m_L}{M} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h_n}{h} \right)$  |                   |  |
| Moment of Inertia of Solidified Fluid            | $\frac{I_s}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left[ \frac{1}{12} \left( \frac{h}{a} \right)^2 + \frac{1}{4} \right]$   | (fig. 4-12)       |  |
| Moment of Inertia of Disc ( $c_d = 0$ )          | $\frac{I_d}{m_L h^2} = 8 \left( \frac{a}{h} \right)^2 \sum_{n=0}^{\infty} \frac{[1 - (2/\kappa_n) \tanh (\kappa_n/2)]}{(\epsilon_n^2 - 1) \epsilon_n}$                            |                   |  |
| Moment of Inertia of Disc ( $c_d \neq 0$ )       | $\frac{I_d}{m_L h^2} = \frac{I_s - \bar{I}}{m_L h^2} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   | (fig. 4-13)       |  |
| Moment of Inertia of Fixed Mass                  | $\frac{I}{m_L h^2} = \frac{I_s}{m_L h^2} - \frac{I_d}{m_L h^2} - \frac{M}{m_L} \left( \frac{H}{h} \right)^2 - \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{h_n}{h} \right)^2$ |                   |  |

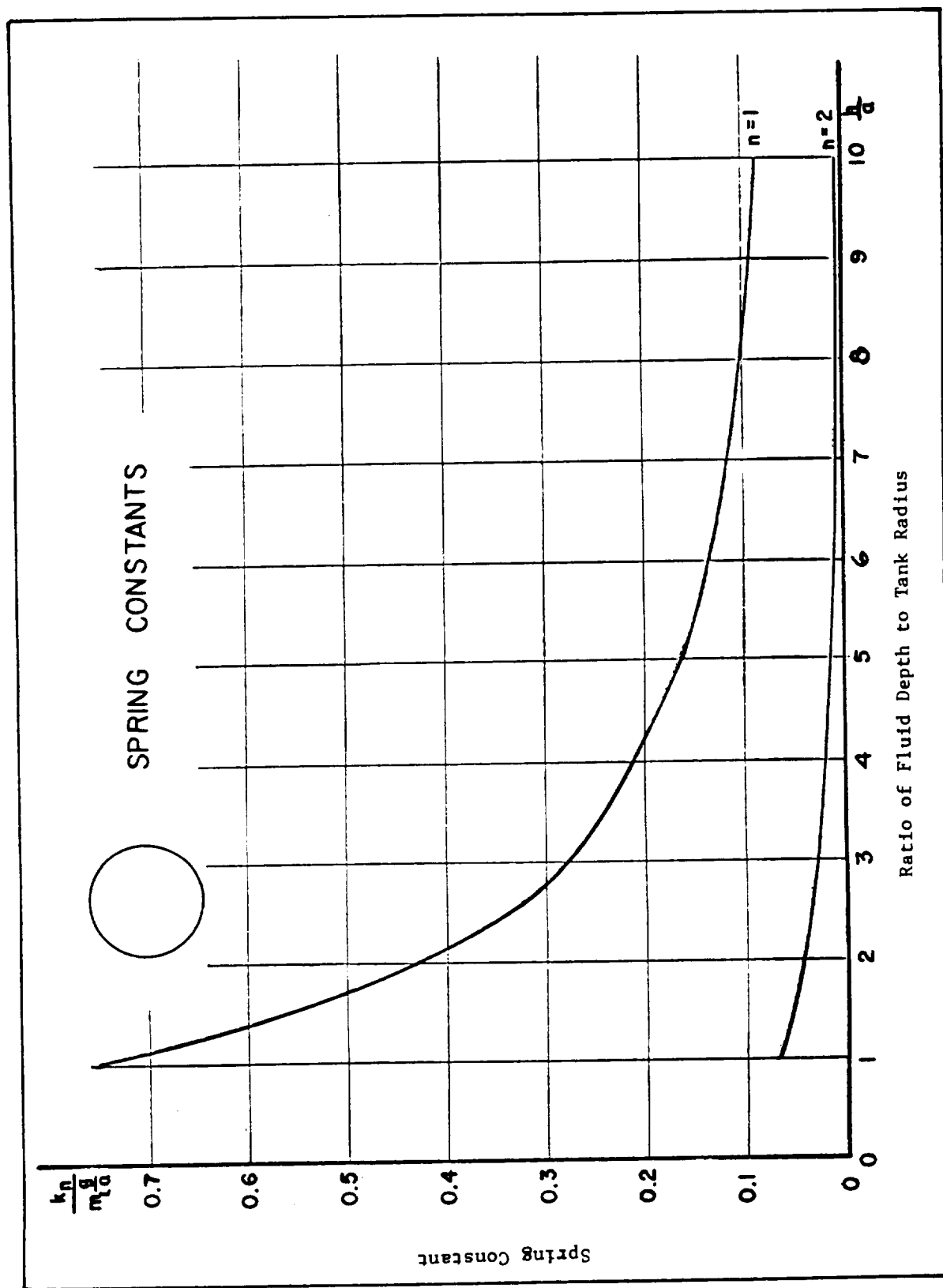


Figure 4-2. Model Element Graph

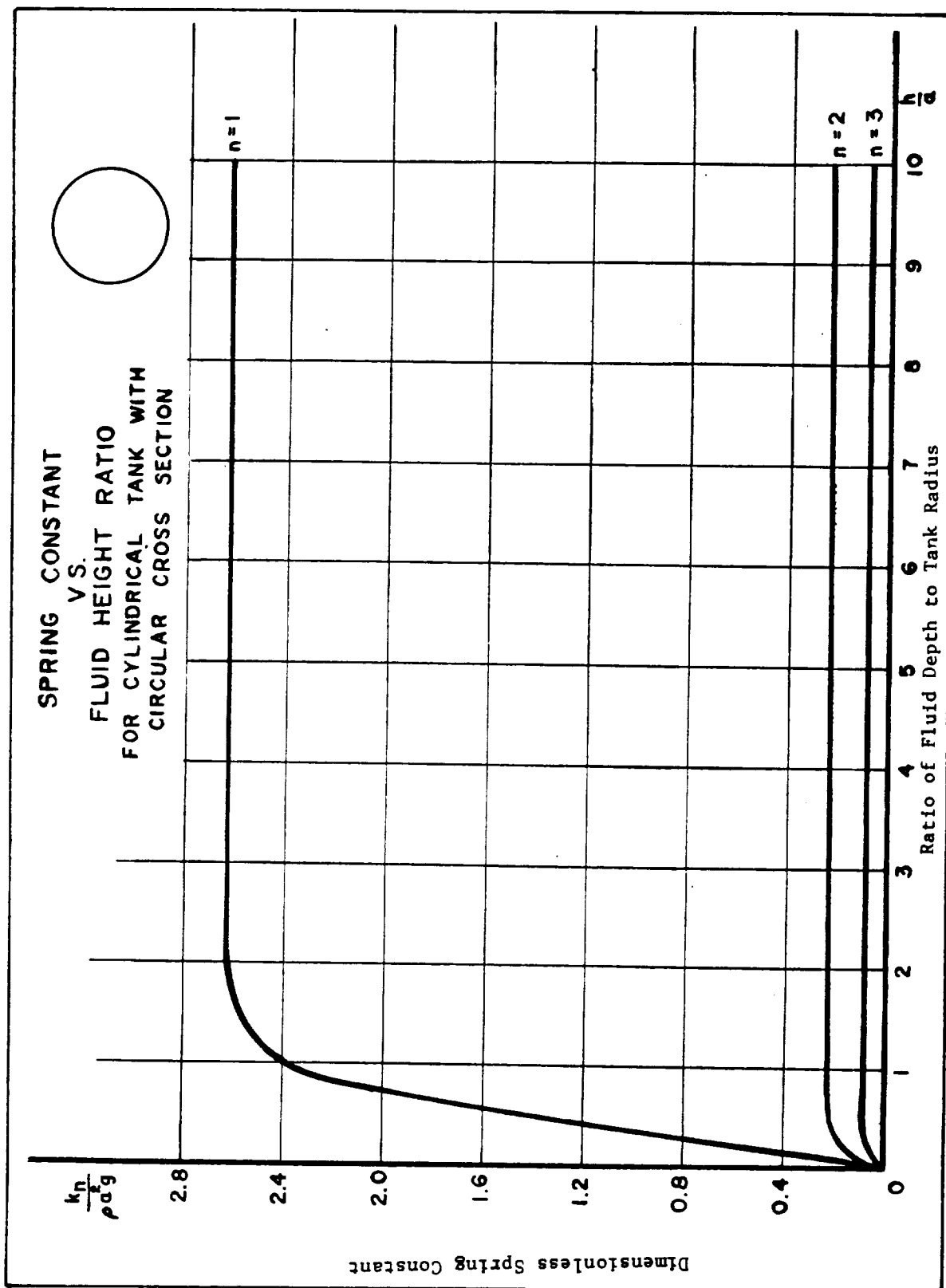


Figure 4-3 Model Element Graph



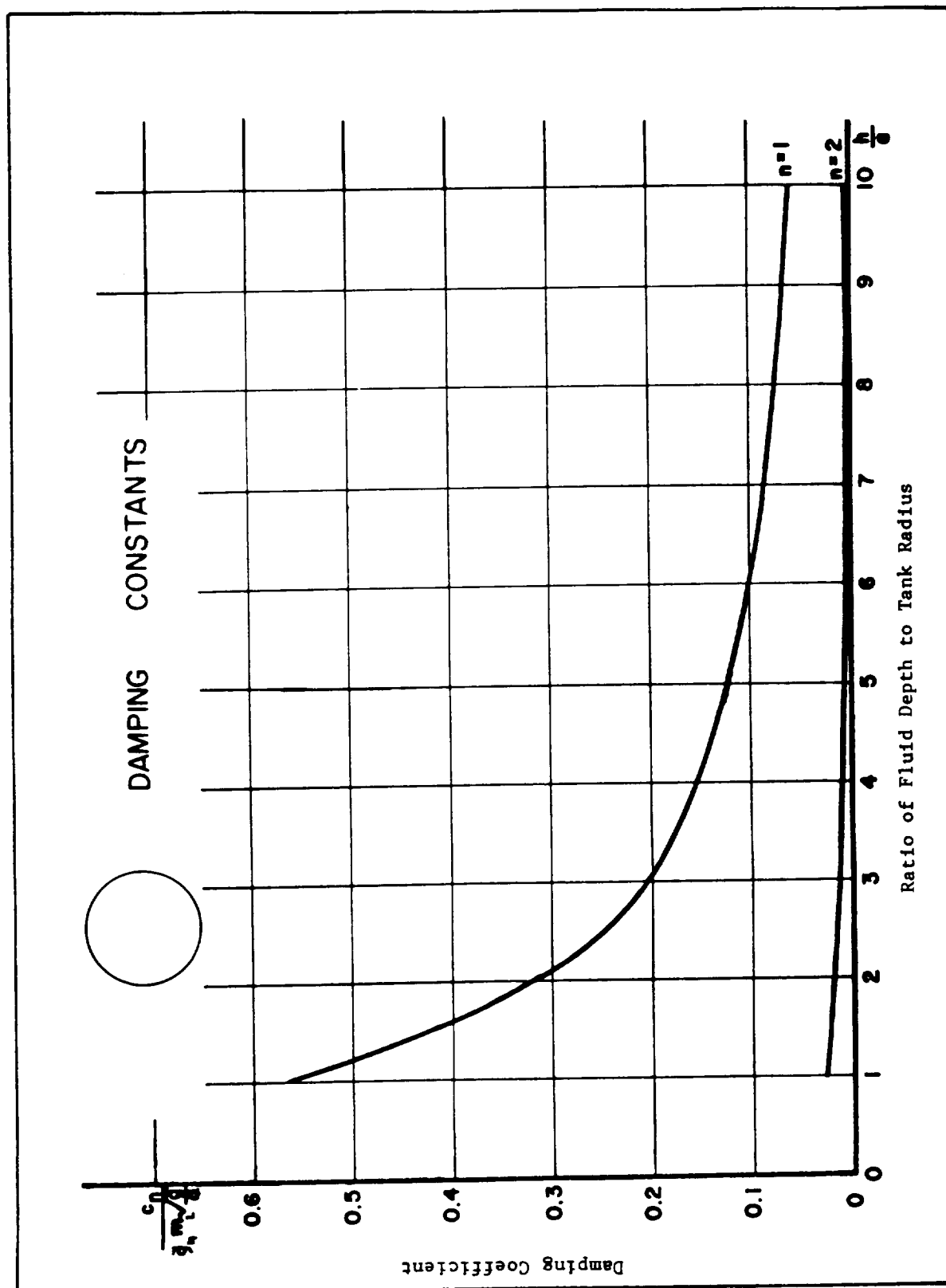


Figure 4-4. Model Element Graph

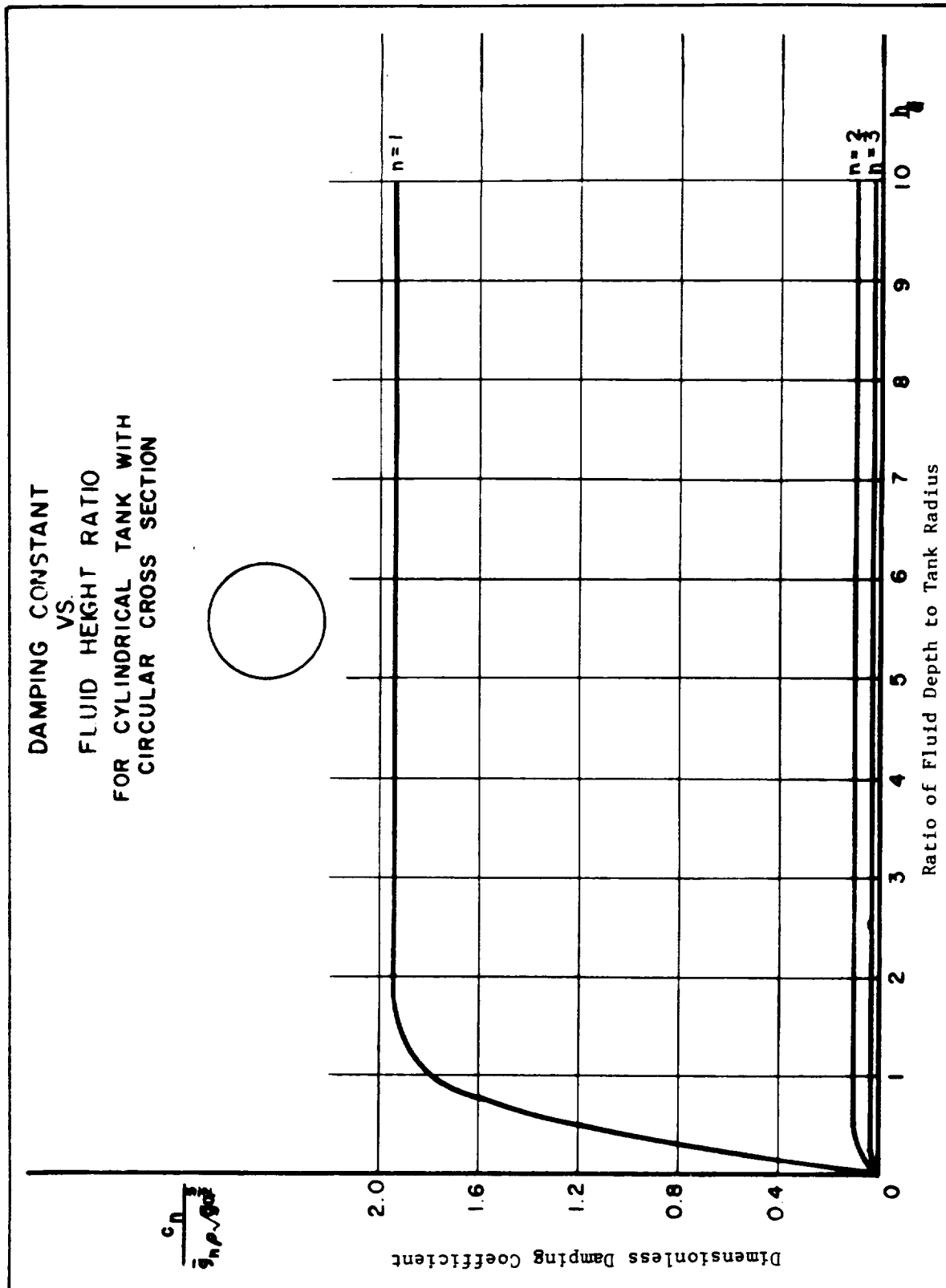


Figure 4-5. Model Element Graph

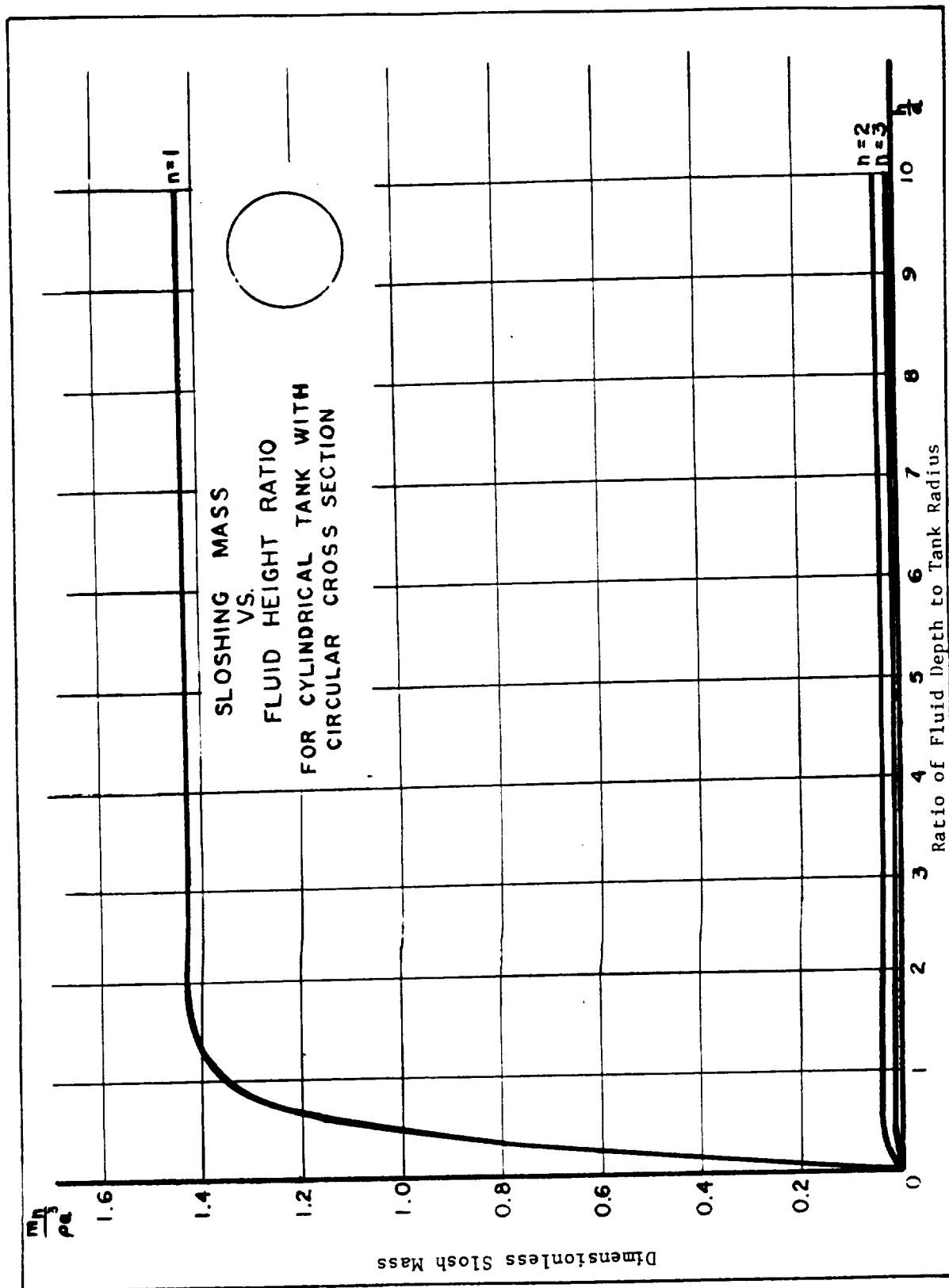


Figure 4-6. Model Element Graph

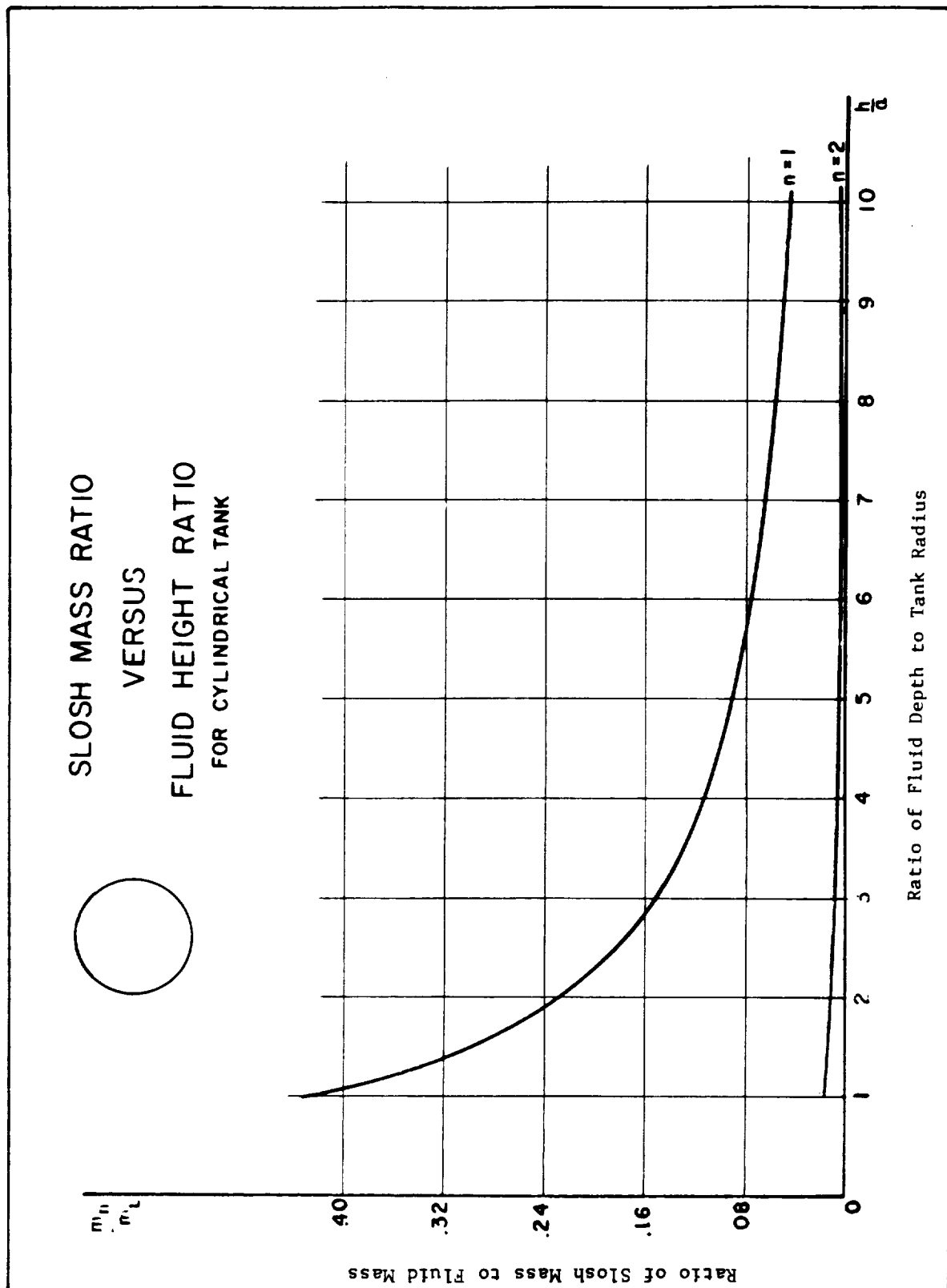


Figure 4-7. Model Element Graph

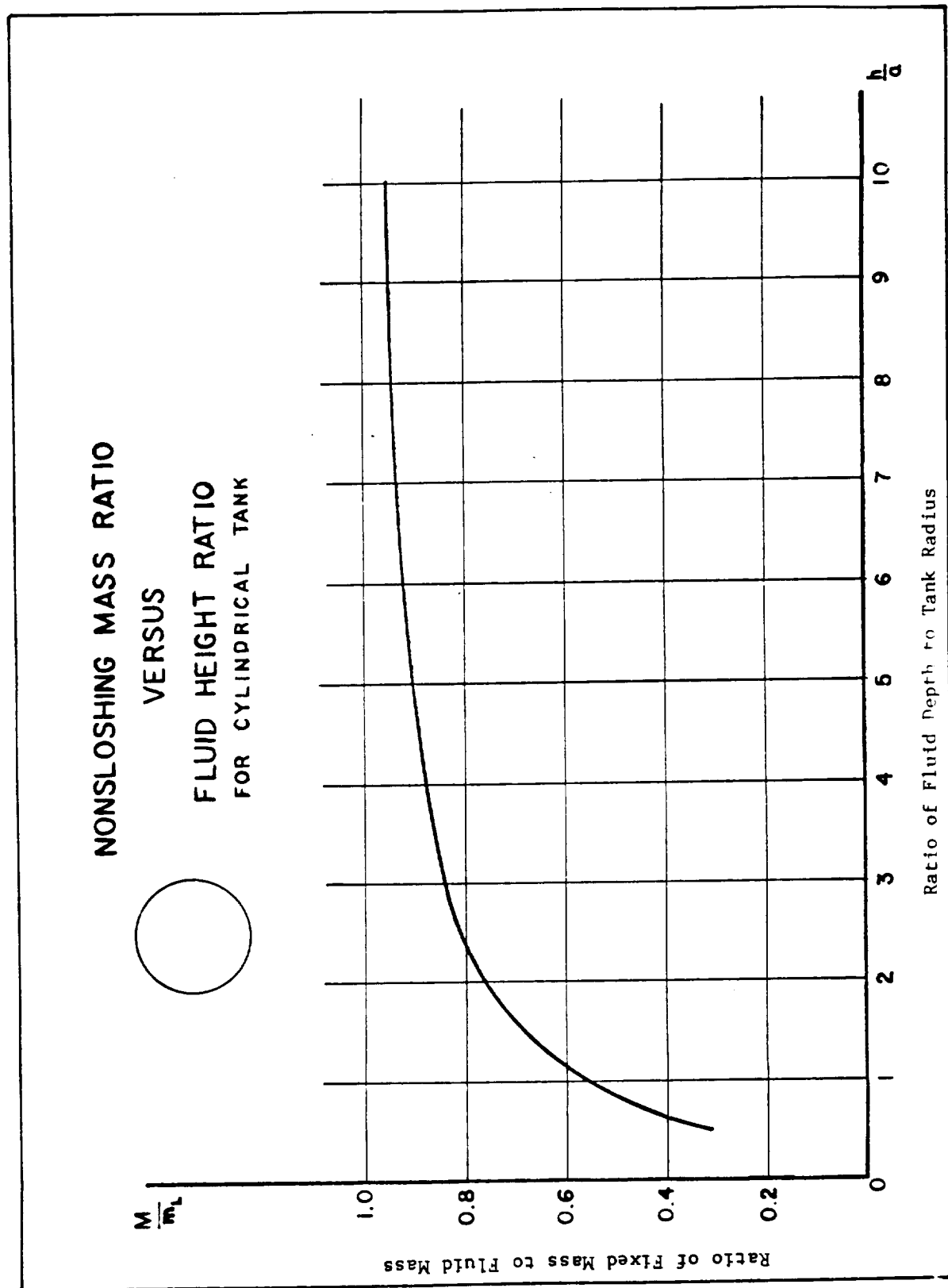


Figure 4-8. Model Element Graph

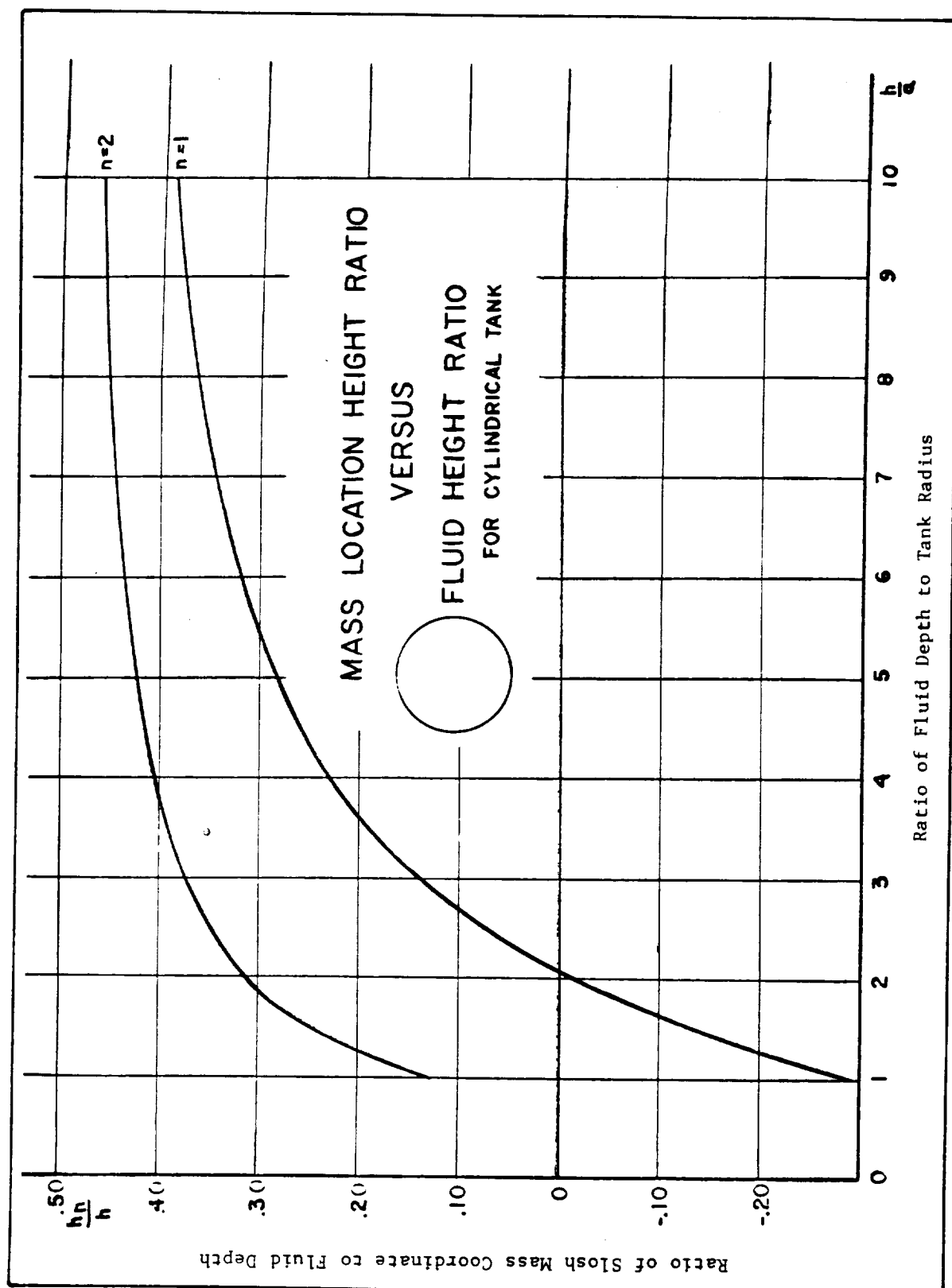


Figure 4-9. Model Element Graph

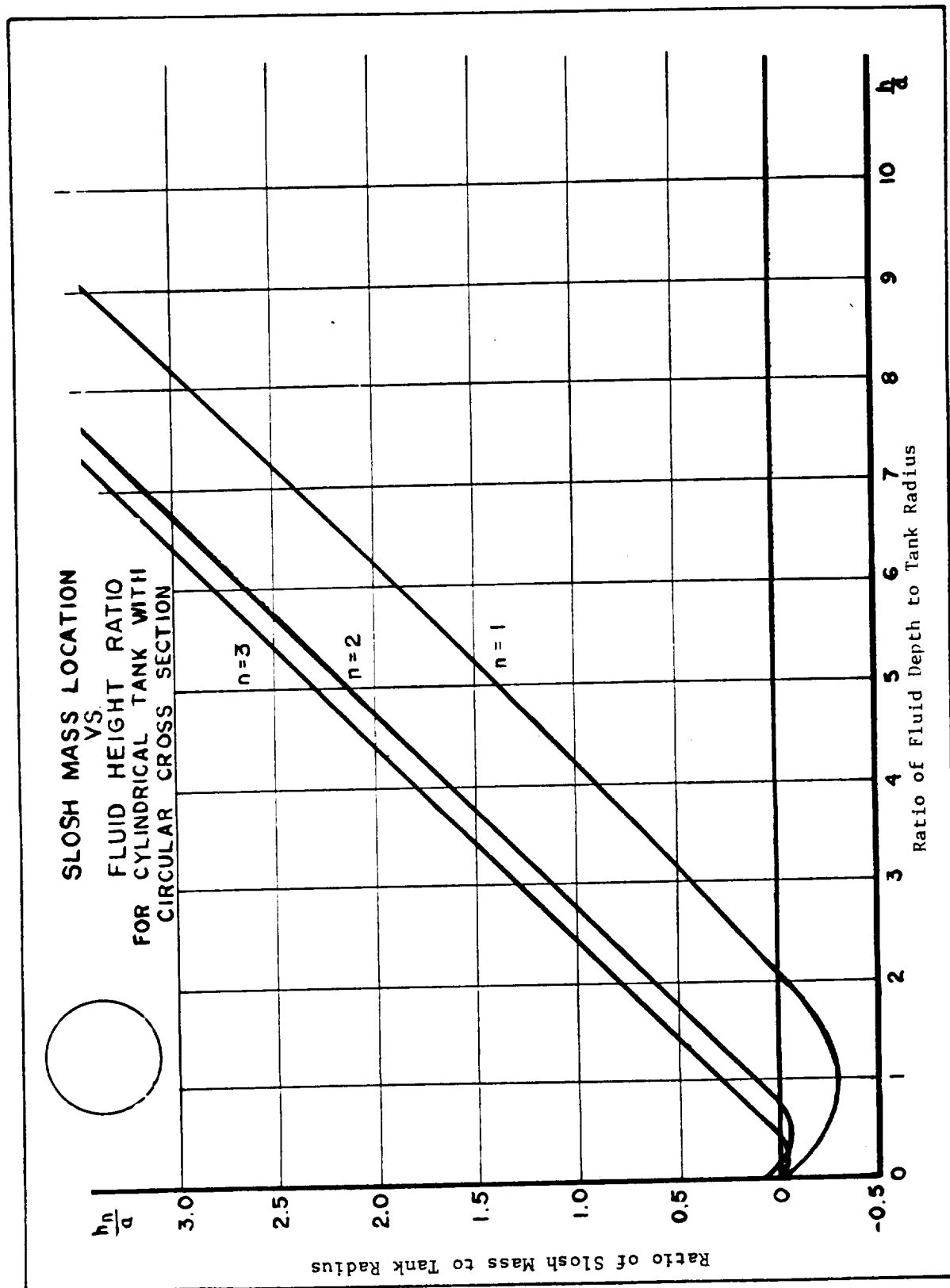


Figure 4-10. Model Element Graph

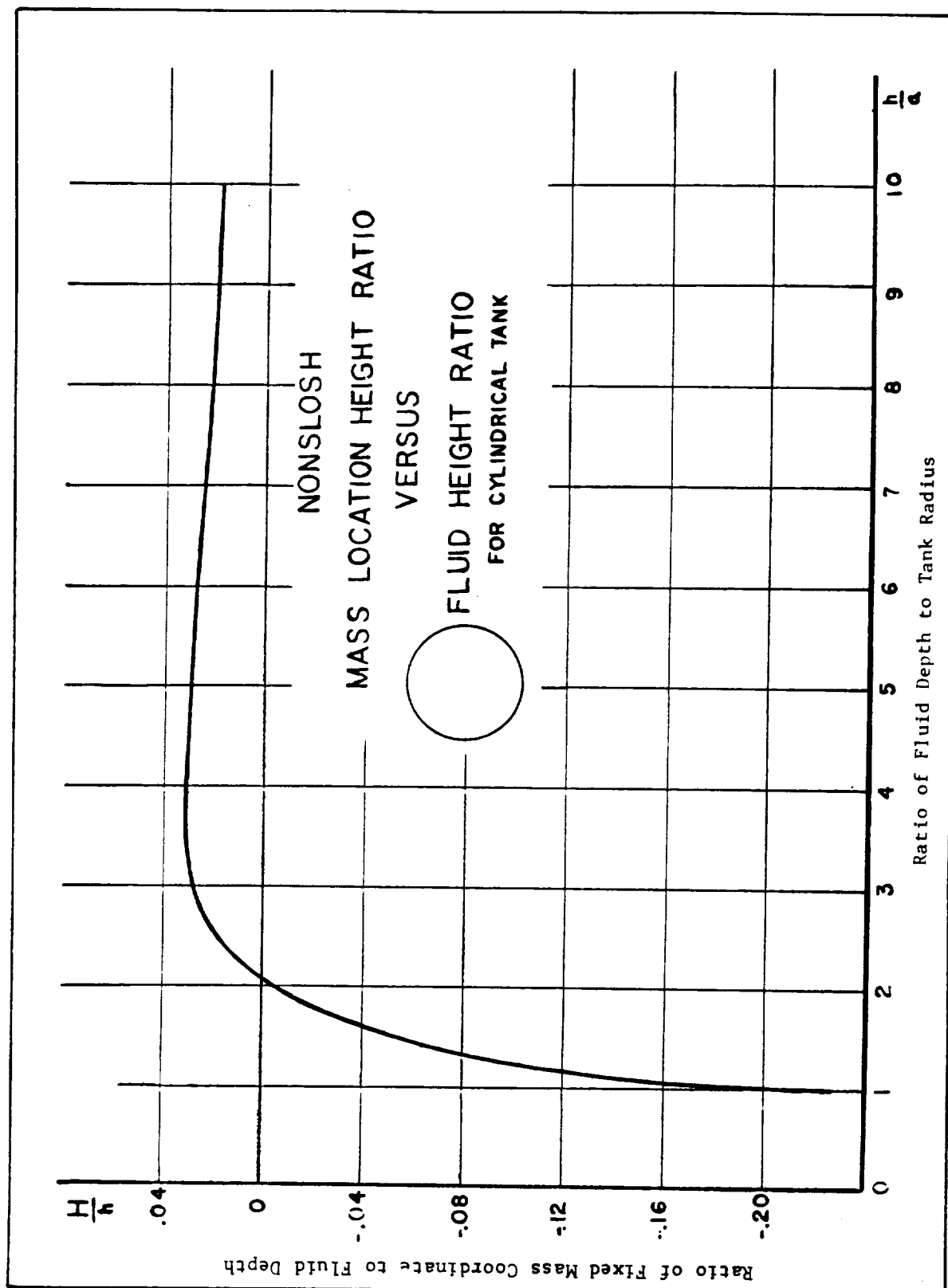


Figure 4-11. Model Element Graph



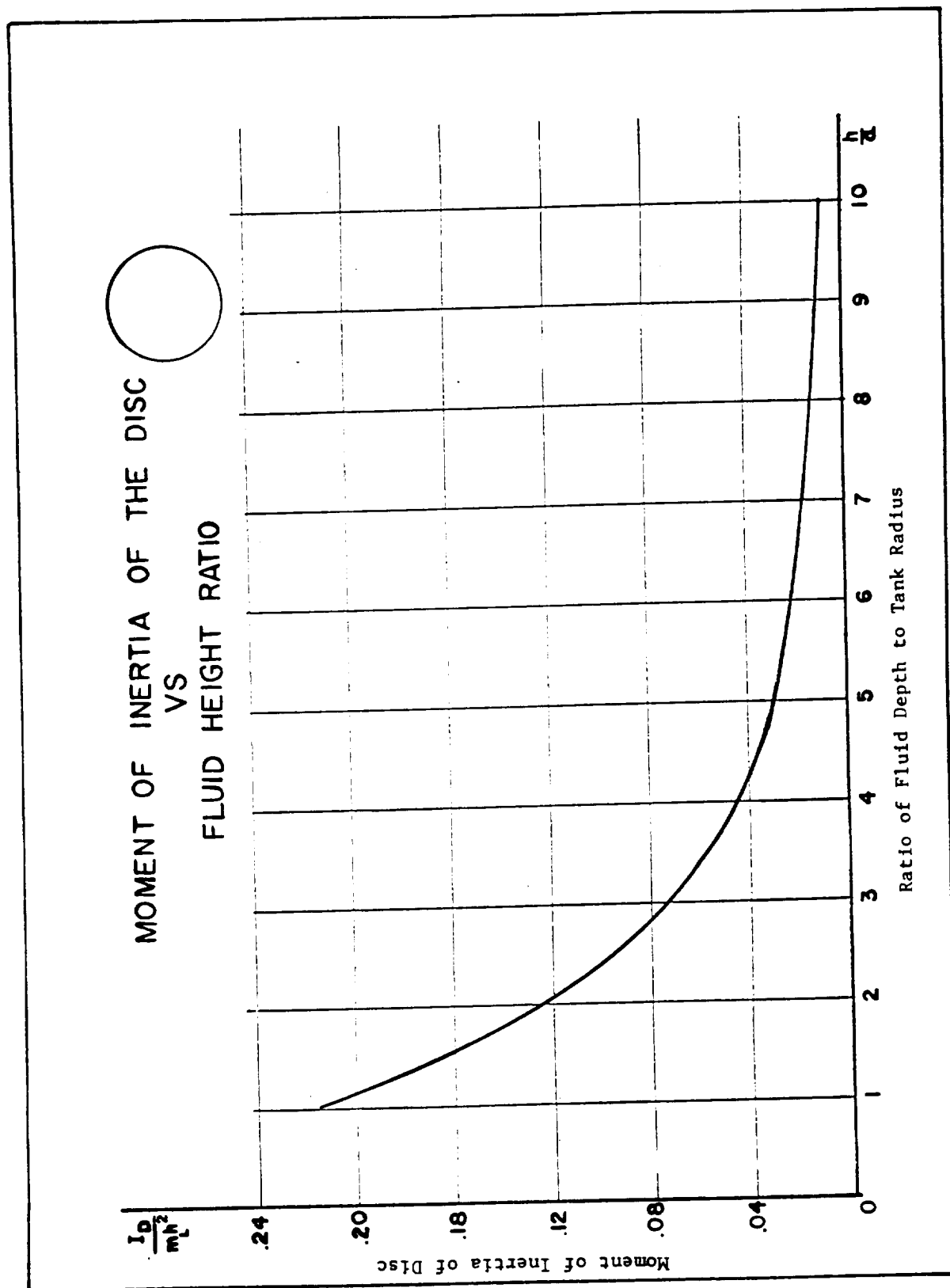


Figure 4-12. Model Element Graph

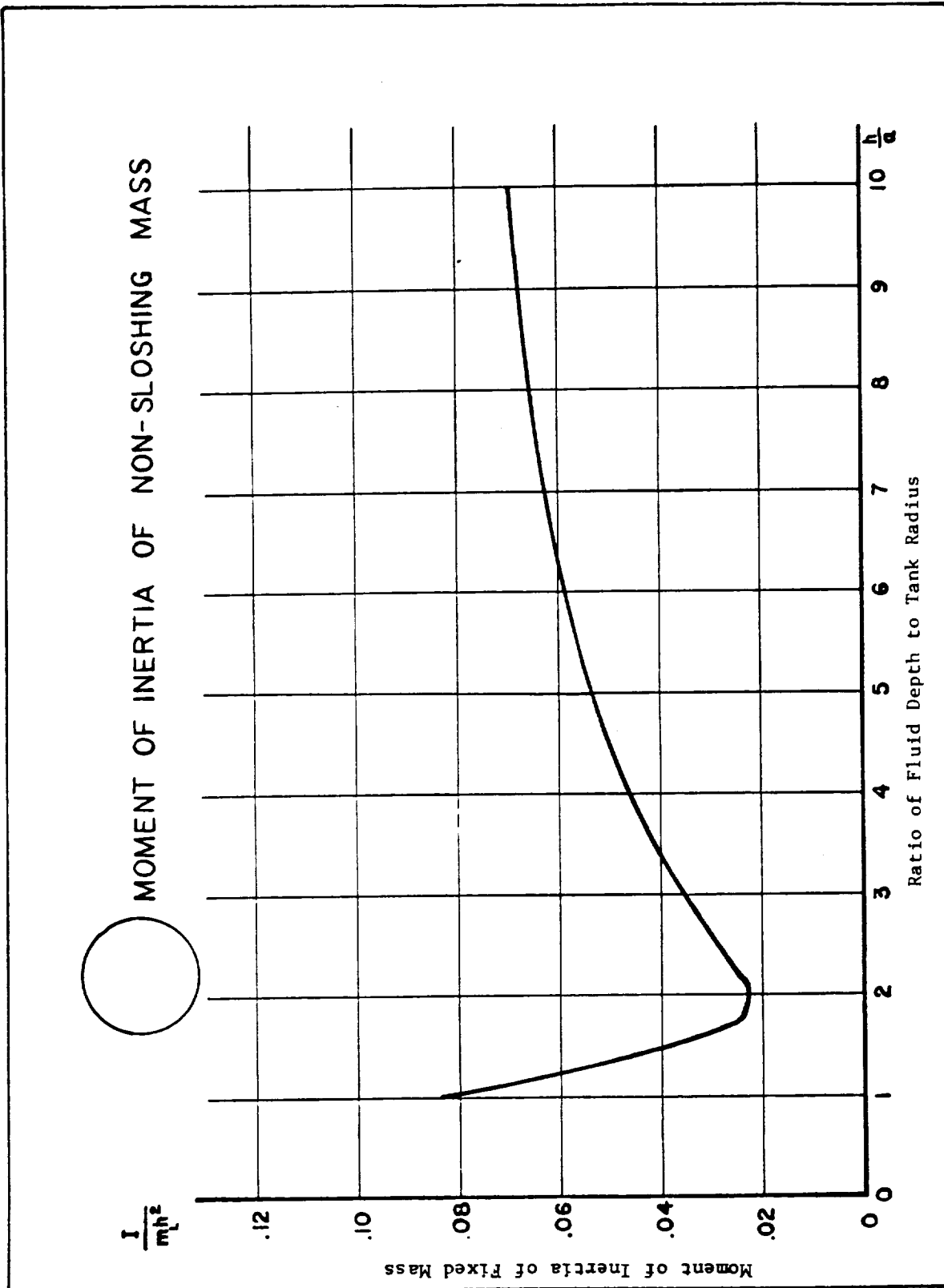


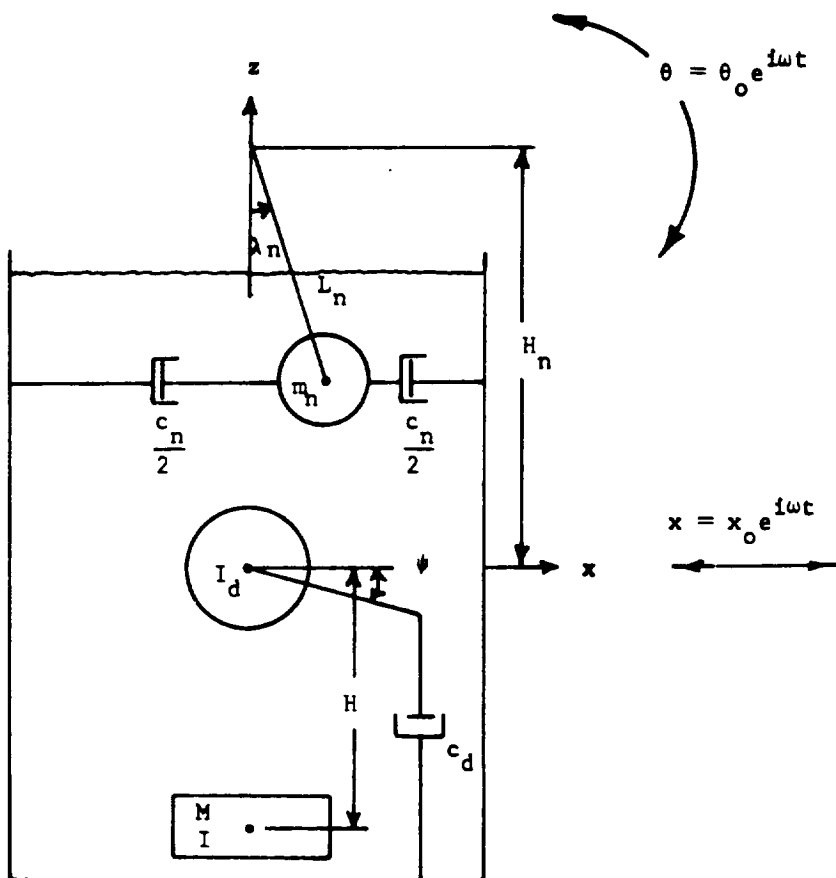
Figure 4-13. Model Element Graph

Table 4-6. Model Analysis

| Circular Cylindrical Tank  | Pendulum Model |
|--|----------------|
| Excitation: Harmonic Translation and/or Pitching   |                |
| <p>Figure 4-14 shows a diagram of the pendulum model used in representing the dynamic response of a liquid in a circular cylindrical tank when subjected to harmonic translation in the x-direction and/or pitching about the y-axis.</p> <p><u>Coordinate System:</u></p> <p>The origin is located at the center of gravity of the undisturbed liquid.</p> <p><u>Model Description:</u></p> <p>The components of the system are as follows:</p> <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the z-axis at a distance <math>H</math> below the coordinate origin.</li> <li>2. A set of movable masses <math>m_n</math> is distributed along the z-axis when the tank is at rest. These modal masses are pendulums having massless lever arms of length <math>L_n</math> attached to the z-axis at distances <math>H_n</math> above the origin. They are constrained by dashpots having viscous damping coefficients <math>c_n</math> to remain approximately in the xy-plane and to move approximately parallel to the x-axis. Angular displacements of the pendulum with respect to the tank (z-axis) are denoted by <math>\lambda_n</math>.</li> <li>3. A massless disc having a moment of inertia <math>I_d</math> is located at the coordinate origin. Its motion is confined to rotation about the y-axis and is cushioned by a dashpot having a viscous damping coefficient <math>c_d</math>. The angular displacement of the disc relative to the tank is defined by <math>\psi</math>.</li> </ol> <p><u>Equations of Motion:</u></p> <p>The equations, obtained through Lagrange's equations, are as follows:</p> <ol style="list-style-type: none"> <li>1. Force Equation: <math display="block">F_x = -M(\ddot{x} + H\ddot{\theta}) - \sum_{n=1}^{\infty} m_n \left[ \ddot{x} + L_n \ddot{\lambda}_n + (H_n - L_n) \ddot{\theta} \right]</math> </li> <li>2. Moment Equation: <math display="block">M_y = - (I + MH^2) \ddot{\theta} - I_d (\ddot{\theta} + \ddot{\psi}) + g \sum_{n=1}^{\infty} m_n L_n \lambda_n - \sum_{n=1}^{\infty} m_n (H_n - L_n) \cdot \left[ L_n \ddot{\lambda}_n + (H - L_n) \ddot{\theta} \right]</math> </li> </ol> |                |

Figure 4-6. Model Analysis (continued)

| Circular Cylindrical Tank   | Pendulum Model |
|---|----------------|
| Excitation: Harmonic Translation and/or Pitching  |                |
| <p data-bbox="204 464 683 495"><u>Equations of Motion (continued):</u></p> <p data-bbox="282 525 548 556">3. Disc Equation:</p> $I_d (\ddot{\theta} + \ddot{\psi}) + c_d \dot{\psi} = 0$ <p data-bbox="282 640 634 672">4. Slosh-Mass Equation:</p> $m_n \left[ \ddot{x} + L_n \ddot{\lambda}_n + (H_n - L_n) \ddot{\theta} \right] + m_n \bar{g}_n \omega_n L_n \dot{\lambda}_n - m_n g \theta = 0$ <p data-bbox="204 787 1341 846">From these equations, the model force in the x-direction and the moment about the y-axis can be found (see Table 4-7).</p> |                |



Circular Cylindrical Tank

Pendulum Model

Excitation: Harmonic Translation and/or Pitching

Figure 4-14. Equivalent Mechanical Model

Table 4-7. Model Force and Moment Resultants

| Circular Cylindrical Tank   | Pendulum Model |
|---|----------------|
| Excitation: Harmonic Translation and/or Pitching  |                |
| Translation in the x-direction, $x_o e^{i\omega t}$ .   |                |
| 1. $F_x = m_L \omega^2 x_o e^{i\omega t} \left[ 1 + \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) \right]$   |                |
| 2. $M_y = m_L h \omega^2 x_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{H_n - L_n}{h} + \frac{g}{h \omega^2} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$  |                |
| Pitching about the y-axis, $\theta_o e^{i\omega t}$ .   |                |
| 1. $F_x = -m_L h \omega^2 \theta_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{H_n - L_n}{h} + \frac{g}{h \omega^2} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$  |                |
| 2. $M_y = -m_L h^2 \omega^2 \theta_o e^{i\omega t} \left[ \frac{1}{m_L h^2} + \frac{M I^2}{m_L h^2} + \frac{I_d}{m_L h^2} \left( 1 - \frac{\omega^2 I_d^2}{c_d^2 + \omega^2 I_d^2} \right) + \sum_{n=0}^{\infty} \frac{m_n (H_n - L_n)^2}{m_L h^2} \right. \\ \left. + \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{H_n - L_n}{h} + \frac{g}{h \omega^2} \right)^2 \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) \right] + i \omega \theta_o e^{i\omega t} \left( \frac{c_d \omega^2 I_d^2}{c_d^2 + \omega^2 I_d^2} \right)$ |                |

Table 4-8. Model Elements

| Circular Cylindrical Tank                        |   | Pendulum Model |  |
|--|---|----------------|--|
| Excitation: Harmonic Translation and/or Pitching |   |                |  |
| Natural Frequency                                | $\omega_n^2 = \frac{g}{a} \epsilon_n \tanh \kappa_n$ (fig. 4-15)  |                |  |
| Pendulum Length                                  | $L_n = g/\omega_n^2$  |                |  |
| Damping Coefficient of Slosh Mass                | $c_n = m_n \bar{g}_n \omega_n$  |                |  |
| Damping Coefficient of Disc                      | $c_d = \bar{c} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   |                |  |
| Ratio of Slosh Mass to Fluid Mass                | $\frac{m_n}{m_L} = \frac{2 \tanh (\kappa_n)}{(\epsilon_n^2 - 1) \kappa_n}$  |                |  |
| Ratio of Fixed Mass to Fluid Mass                | $\frac{M}{m_L} = 1 - \sum_{n=0}^{\infty} \frac{m_n}{m_L}$ (fig. 4-16, 4-17)   |                |  |
| Ratio of Pendulum Mass Coordinate to Fluid Depth | $\left  \frac{H_n - L_n}{h} \right  = \frac{1}{2} \left[ 1 - \frac{4}{\kappa_n} \tanh \frac{\kappa_n}{2} \right]$   |                |  |
| Ratio of Fixed Mass Coordinate to Fluid Depth    | $\left  \frac{H}{h} \right  = \frac{m_L}{M} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{H_n - L_n}{h} \right)$  |                |  |
| Moment of Inertia of Solidified Fluid            | $\frac{I_s}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left[ \frac{1}{12} \frac{h}{a}^2 + \frac{1}{4} \right]$  |                |  |
| Moment of Inertia of Disc ( $c_d = 0$ )          | $\frac{I_d}{m_L h^2} = 8 \left( \frac{a}{h} \right)^2 \sum_{n=0}^{\infty} \frac{[1 - (2/\kappa_n) \tanh (\kappa_n/2)]}{(\epsilon_n^2 - 1) \epsilon_n}$                                  |                |  |
| Moment of Inertia of Disc ( $c_d \neq 0$ )       | $\frac{I_d}{m_L h^2} = \frac{I_s - \bar{I}}{m_L h^2} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   |                |  |
| Moment of Inertia of Fixed Mass                  | $\frac{I}{m_L h^2} = \frac{I_s}{m_L h^2} - \frac{I_d}{m_L h^2} - \frac{M}{m_L} \left( \frac{H}{h} \right)^2 - \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left[ \frac{H_n - L_n}{h} \right]^2$ |                |  |

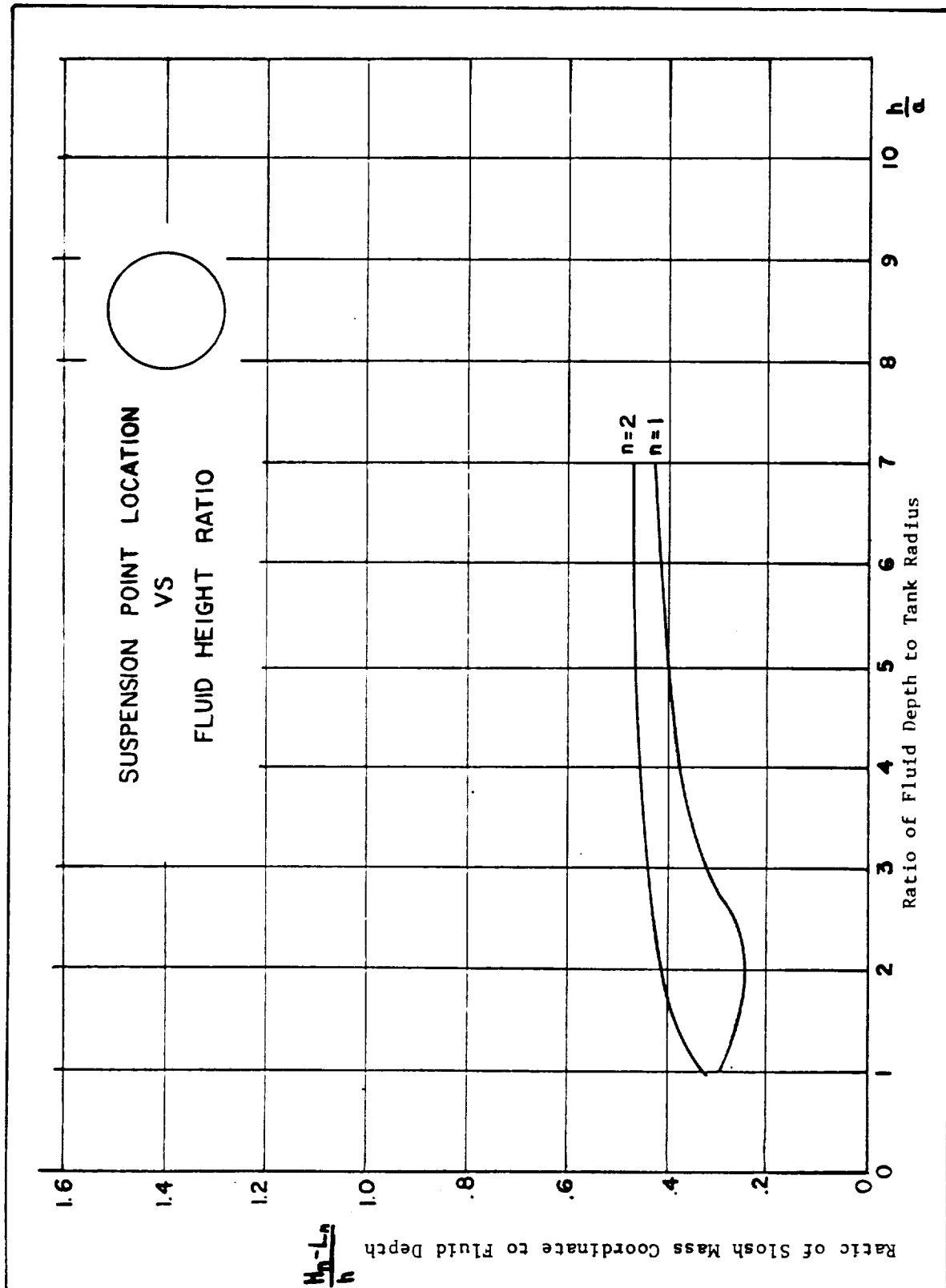


Figure 4-15. Model Element Graph



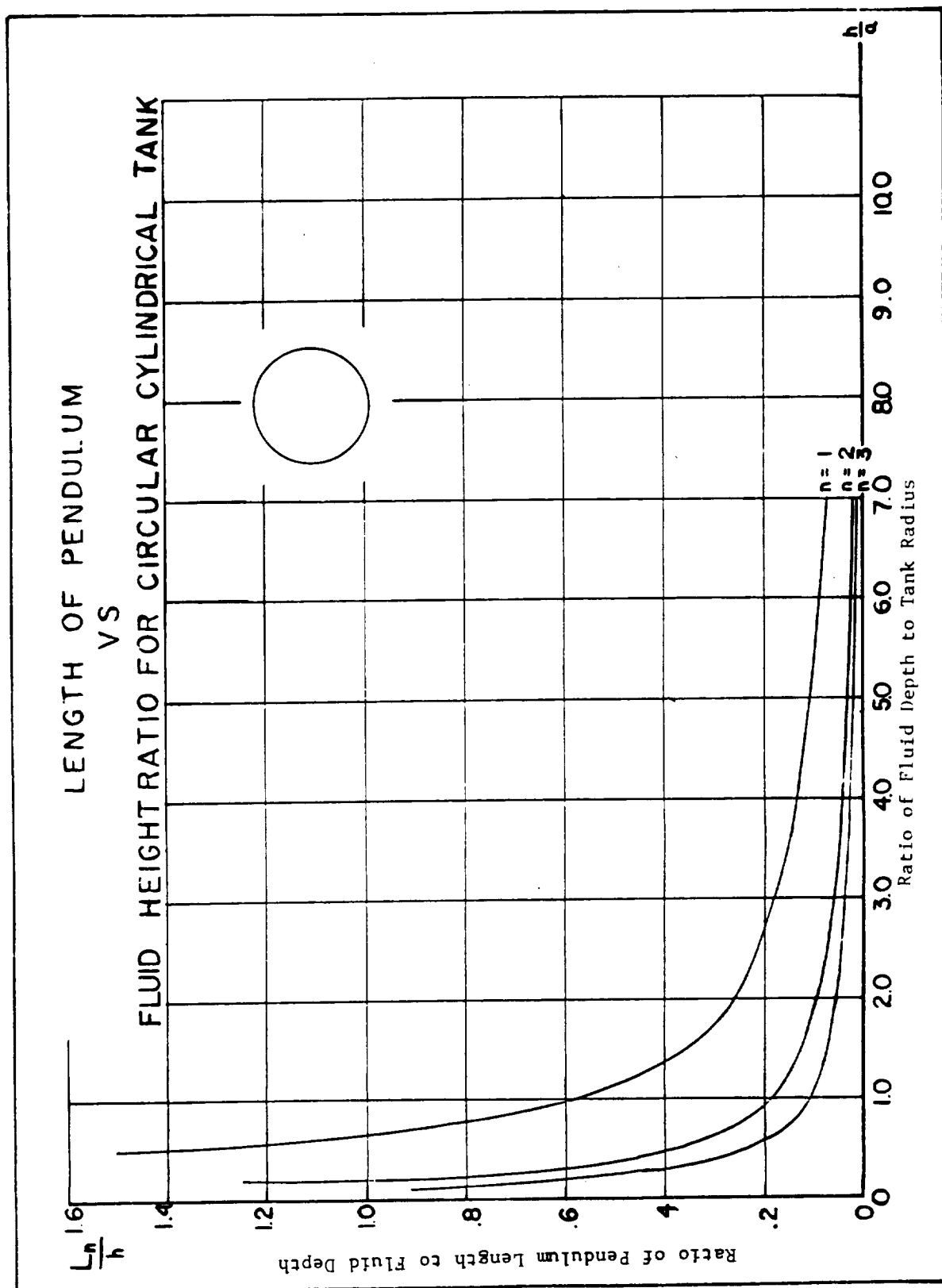


Figure 4-16. Model Element Graph

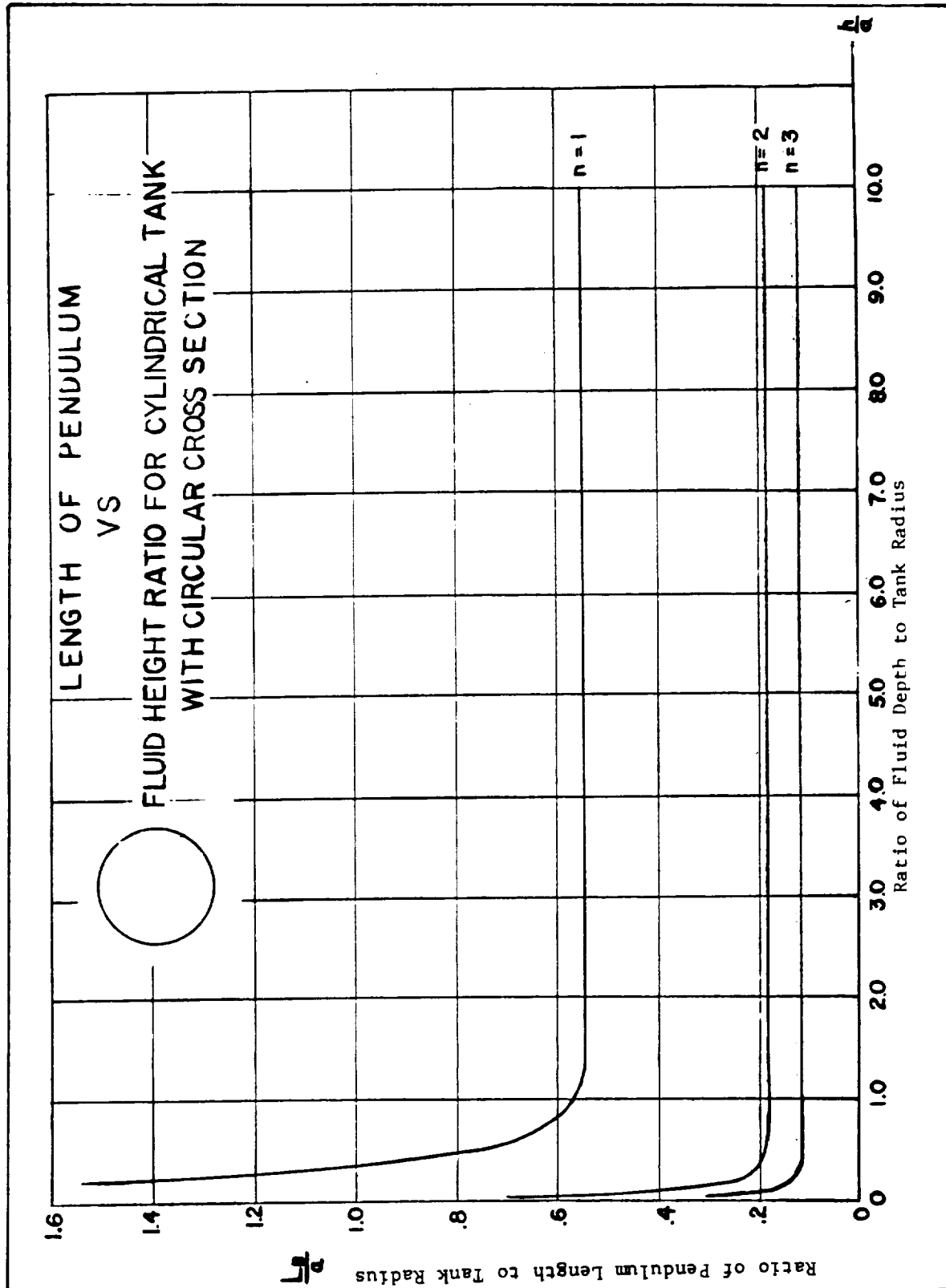


Figure 4-17. Model Element Graph

Table 4-9. Model Analysis

| Circular Cylindrical Tank  | Pendulum Model |
|--|----------------|
| Excitation: Arbitrary Translation and/or Pitching  |                |
| <p>Figure 4-18 shows a diagram of the pendulum model used in representing the dynamic response of a liquid in a circular cylindrical tank when subjected to an arbitrary translation in the x-direction and/or pitching about the y-axis.</p> <p><u>Coordinate System:</u></p> <p>The origin is located at the geometric center of the tank bottom.</p> <p><u>Model Description:</u></p> <p>The components of the system are as follows:</p> <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the z-axis at a distance <math>H</math> above the coordinate origin.</li> <li>2. A set of movable masses <math>m_n</math> is distributed along the z-axis when the tank is at rest. The modal masses are pendulums having massless lever arms of length <math>L_n</math> attached to the z-axis at distances <math>H_n</math> above the origin. Angular displacements of the pendulums with respect to the tank (z-axis) is denoted by <math>\lambda_n</math> and <math>\theta</math> is a space fixed coordinate defining the pitching angle of the container. Angular displacements of the pendulums are therefore defined by a generalized space fixed coordinate, <math>\Gamma_n = \lambda_n - \theta</math>.</li> </ol> <p><u>Equations of Motion:</u></p> <p>The equations, obtained through Lagrange's equations, are as follows:</p> <ol style="list-style-type: none"> <li>1. Force Equation: <math display="block">F = - (M + m_n) \ddot{x} - [MH + m_n(H_n - L_n)] \ddot{\theta} - m_n L_n \ddot{\lambda}_n</math> </li> <li>2. Moment Equation: <math display="block">M = - [MH + m_n(H_n - L_n)] \ddot{x} - [MH^2 + m_n(H_n - L_n)^2 + I] \ddot{\theta} + [MH + m_n(H_n - L_n)] \theta - m_n L_n (H_n - L_n) \ddot{\lambda}_n + m_n g L_n \lambda_n</math> </li> <li>3. Slosh Mass Equation: <math display="block">\ddot{\lambda}_n + \omega_n^2 \lambda_n = - \frac{1}{L_n} \left[ \ddot{x} + (H_n - L_n) \ddot{\theta} - g \theta \right]</math> </li> </ol> <p>Using the method of Laplace transforms, the force and moment have been represented as transformed variables, which thereby become a function of the Laplace-transform operator <math>s</math> (see Table 4-10).</p> |                |

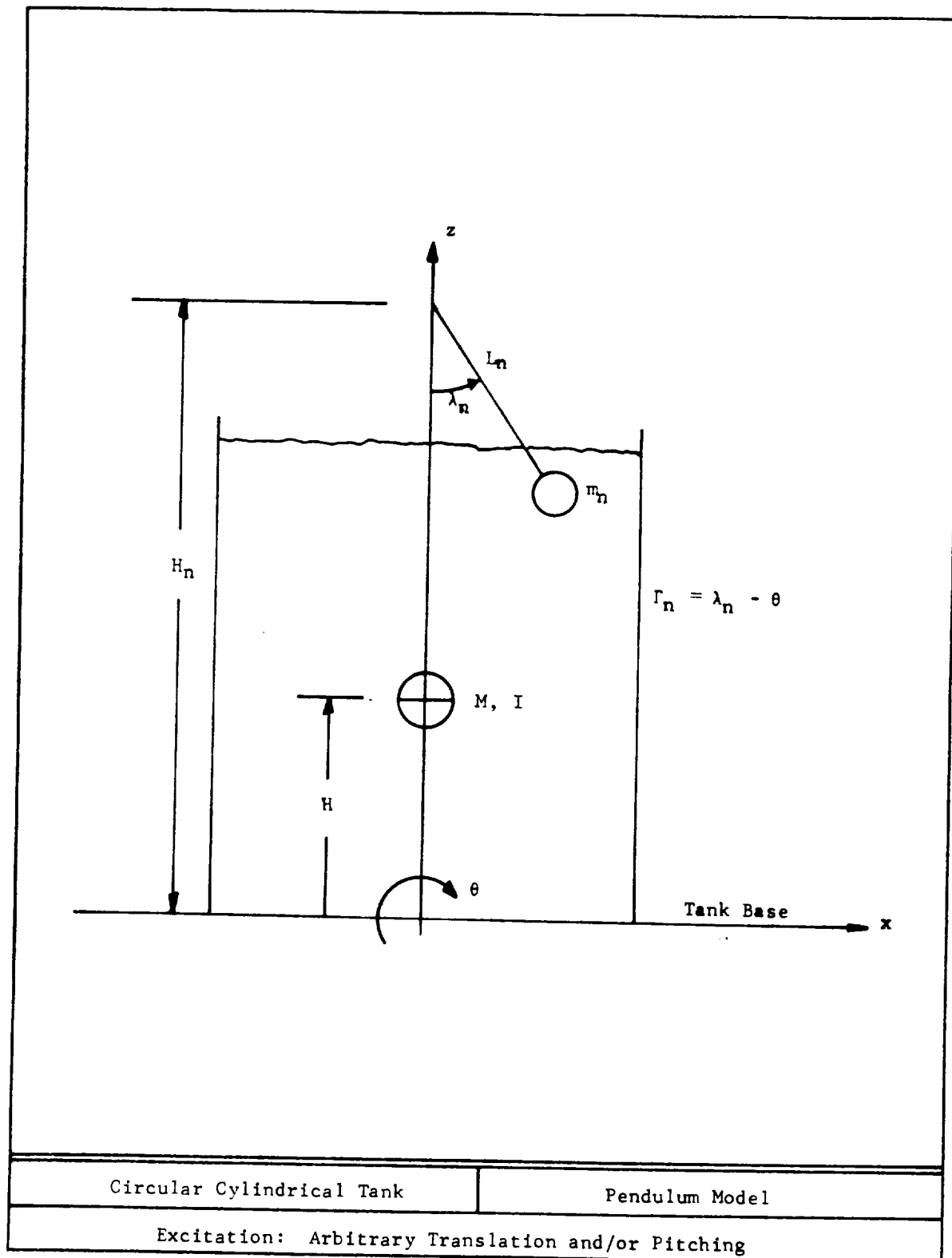


Figure 4-18. Equivalent Mechanical Model

Table 4-10. Model Force and Moment Resultants

| Circular Cylindrical Tank   | Pendulum Model |
|---|----------------|
| Excitation: Arbitrary Translation and/or Pitching   |                |
| Arbitrary translation in the x'-direction and/or pitching about the y'-axis.  |                |
| 1. $F(s) = -Mx(s)s^2 - M\ddot{\theta}(s)s^2 - gs^2 \sum_{n=0}^{\infty} m_n \left( \sum_{n=0}^{\infty} \frac{x(s) + H_n \theta(s)}{L_n(s^2 + \omega_n^2)} \right)$   |                |
| 2. $M(s) = -M\ddot{x}(s)s^2 - (I + M\ddot{\theta}(s)s^2 + g \sum_{n=0}^{\infty} \left[ (M\ddot{\theta}(s) + m_n H_n) \theta(s) - m_n H_n \frac{[x(s) + H_n \theta(s)] s^2}{L_n(s^2 + \omega_n^2)} \right])$ |                |



Table 4-11. Model Elements

| Circular Cylindrical Tank                         | Pendulum Model   |
|---|--|
| Excitation: Arbitrary Translation and/or Pitching |  |
| Natural Frequency                                 | $\omega_n^2 = \frac{g}{a} \epsilon_n \tanh \kappa_n$   |
| Pendulum Length                                   | $L_n = \frac{g}{\omega_n^2}$   |
| Slosh Mass  | $m_n = m_L \left[ \frac{2 \tanh \kappa_n}{\kappa_n (\epsilon_n^2 - 1)} \right]$  |
| Fixed Mass  | $M = m_L - \sum_{n=1}^{\infty} m_n$  |
| Slosh Mass Coordinate                             | $ H_n - L_n  = h \left[ 1 - \frac{2 - \cosh \kappa_n}{\kappa_n \sinh \kappa_n} \right]$  |
| Fixed Mass Coordinate                             | $ H  = \frac{m_L h}{m_L - m_n} \left[ \frac{1}{2} + \frac{a^2}{4h^2} - \frac{2(2 + \kappa_n \sinh \kappa_n - \cosh \kappa_n)}{\kappa_n^2 (\epsilon_n^2 - 1) \cosh \kappa_n} \right]$ |
| Moment of Inertia of Fixed Mass                   | $I = m_L h^2 \left[ \frac{1}{3} - \frac{8\kappa_n + (2\kappa_n^2 - 8)\sinh \kappa_n - 2\kappa_n \cosh \kappa_n}{\kappa_n^3 (\epsilon_n^2 - 1) \cosh \kappa_n} \right] - MH^2$        |





## IV. Rigid Tanks

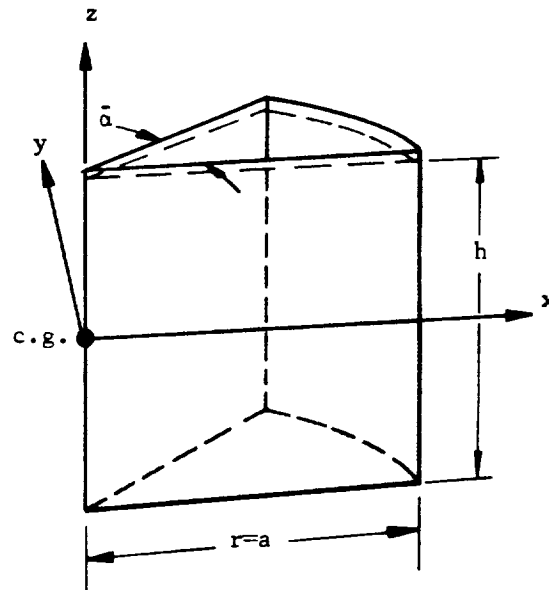
## 4.3 Sector Tank

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## IV. Rigid Tanks

## 4.3 Sector Tank



Container: The tank is a sector of a right circular cylinder of radius  $a$  and filled with a liquid to a depth  $h$ . The vertex angle is denoted by  $\bar{\alpha}$ .

Coordinate System: The origin is located at the center of gravity of the undisturbed fluid which would be contained in a right circular cylinder generated by revolving the sector tank about the  $z$ -axis. The  $x$ -axis must lie in the sector wall.

References: (4)

Comments: The results given in this section are not valid for  $\bar{\alpha} = \frac{\pi}{2}, \frac{3\pi}{2}$ .

Table 4-12. Boundary Conditions, Velocity Potential, and Natural Frequency

| Sector Tank  |   | Excitation: Harmonic Translation   |
|--|---|--|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .  |   |  |
| 1. Boundary conditions:  |   |  |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = i\omega x_0 e^{i\omega t} \cos \phi$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$  |  |
| (c) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = 0$  | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\alpha} = -i\omega x_0 e^{i\omega t} \sin \alpha$ | (e) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$ |
| 2. Velocity potential:   |   |  |
| $\phi = i\omega x_0 e^{i\omega t} \left[ r \cos \phi + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a b_{mn} J_{mn}(\rho_{mn}) \cos \phi_{mn} \cosh(\kappa_{mn} + \zeta_{mn})}{(n_{mn}^2 - 1)} \right]$ |   |  |
| 3. Natural angular frequency:  |   |  |
| $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$  |   |  |
| where $\epsilon_{mn}$ are roots of $J'_{m/2\alpha}(\epsilon_{mn}) = 0$ .   |   |  |

Table 4-12. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

| Sector Tank  |  | Excitation: Harmonic Translation   |
|--|--|--|
| Translation in the y-direction, $y = y_0 e^{i\omega t}$ .  |  |  |
| 1. Boundary conditions:  |  |  |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = i\omega y_0 e^{i\omega t} \sin \phi$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$   |  |
| (c) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = i\omega y_0 e^{i\omega t}$  | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\bar{\alpha}} = i\omega y_0 e^{i\omega t} \cos \bar{\alpha}$ | (e) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$ |
| 2. Velocity potential:   |  |  |
| $\phi = i\omega y_0 e^{i\omega t} \left[ r \sin \phi + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c_m b_m J_m(\rho_{mn}) \cos \bar{\phi}_m \cosh(\kappa_{mn} + \zeta_{mn})}{(\eta_{mn}^2 - 1)} \right]$ |  |  |
| 3. Natural angular frequency:  |  |  |
| $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$  |  |  |
| where $\epsilon_{mn}$ are roots of $J'_m(2\alpha_{mn}) = 0$ .  |  |  |

Table 4-13. Liquid Force and Moment Resultants

| Sector Tank   | Excitation: Harmonic Translation |
|---|----------------------------------|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ and translation in the y-direction, $y = y_0 e^{i\omega t}$ .   |                                  |
| 1. $F_x = m_L \omega^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2(-1)^{m+1} \begin{pmatrix} a_m \\ c_m \end{pmatrix} \left( \frac{\bar{b}_m \sin \bar{\alpha} \tanh \kappa_{mn}}{\bar{\alpha} a (\eta_{mn}^2 - 1) \kappa_{mn}} \right) \left( \frac{\alpha^2 J_{m/2\alpha}(\epsilon_{mn})}{\pi^2 m^2 - \alpha^2} + L_0(\epsilon_{mn}) \right) \right]$                                       |                                  |
| 2. $F_y = m_L \omega^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2 \begin{pmatrix} a_m \\ c_m \end{pmatrix} \left( \frac{\bar{b}_m [1 - (-1)^m \cos \bar{\alpha}] \tanh \kappa_{mn}}{\bar{\alpha} a (\eta_{mn}^2 - 1) \kappa_{mn}} \right) \left( \frac{\alpha^2 J_{m/2\alpha}(\epsilon_{mn})}{\pi^2 m^2 - \alpha^2} + L_0(\epsilon_{mn}) \right) \right]$                                    |                                  |
| 3. $M_y = m_L a \omega^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t} \left[ \left\{ \frac{1}{4(h/a)} \left( 1 + \frac{(\sin \bar{\alpha} \cos \bar{\alpha})/\bar{\alpha}}{\sin^2 \bar{\alpha}/\bar{\alpha}} \right) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+1} \begin{pmatrix} a_m \\ c_m \end{pmatrix} \left( \frac{\bar{b}_m \sin \bar{\alpha}}{\bar{\alpha} a (\eta_{mn}^2 - 1) \epsilon_{mn}} \right) \right\} \right]$  |                                  |
| 4. $M_x = -m_L a \omega^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t} \left[ \left\{ \frac{\alpha^2 J_{m/2\alpha}(\epsilon_{mn})}{\pi^2 m^2 - \alpha^2} + L_0(\epsilon_{mn}) \right\} \left[ \tanh \kappa_{mn} + \frac{2}{\kappa_{mn}} \left( \frac{1}{\cosh \kappa_{mn}} - 1 \right) \right] + \frac{2\bar{\alpha}^2 \epsilon_{mn} L_2(\epsilon_{mn})}{(\pi^2 m^2 - \alpha^2) \kappa_{mn} \cosh \kappa_{mn}} \right] \right] + \frac{2mga \sin \bar{\alpha}}{3 \bar{\alpha}}$ |                                  |
| 4. $M_x = -m_L a \omega^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t} \left[ \left\{ \frac{1}{4(h/a)} \left( \frac{(2 \sin^2 \bar{\alpha})/\bar{\alpha}}{1 - (\sin \bar{\alpha} \cos \bar{\alpha})/\bar{\alpha}} \right) - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \begin{pmatrix} a_m \\ c_m \end{pmatrix} \left( \frac{\bar{b}_m [1 - (-1)^m \cos \bar{\alpha}]}{\bar{\alpha} a (\eta_{mn}^2 - 1) \epsilon_{mn}} \right) \right\} \right]$                                   |                                  |
| 4. $M_x = -m_L a \omega^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t} \left[ \left\{ \frac{\alpha^2 J_{m/2\alpha}(\epsilon_{mn})}{\pi^2 m^2 - \alpha^2} + L_0(\epsilon_{mn}) \right\} \left[ \tanh \kappa_{mn} + \frac{2}{\kappa_{mn}} \left( \frac{1}{\cosh \kappa_{mn}} - 1 \right) \right] + \frac{2mga(1 - \cos \bar{\alpha})}{3 \bar{\alpha}} \right]$  |                                  |

Table 4-14. Boundary Conditions, Velocity Potential, and Natural Frequency

| Sector Tank   | Excitation: Harmonic Pitching   |
|---|---|
| Pitching about the x-axis, $\chi = \chi_0 e^{i\omega t}$ .  |   |
| 1. <u>Boundary Conditions:</u>  |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = -i\omega z \chi_0 e^{i\omega t} \sin \phi$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = i\omega r \chi_0 e^{i\omega t} \sin \phi$       |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=+h/2} = 0$   | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = -i\omega z \chi_0 e^{i\omega t}$ |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\alpha} = -i\omega z \chi_0 e^{i\omega t} \cos \alpha$  |   |
| 2. Velocity potential:  |   |
| $\phi = -i\omega \chi_0 e^{i\omega t} \left[ rz \sin \phi - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_{m/2\alpha}(\rho_{mn}) \cos \bar{\phi}_m (C_{mn} \cosh \zeta_{mn} + D_{mn} \sinh \zeta_{mn}) \right]$ |   |
| 3. Natural angular frequency:   |   |
| $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$   |   |
| where $\epsilon_{mn}$ are zeros of $J'_{m/2\alpha}(\epsilon_{mn}) = 0$ .  |   |

Table 4-14. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

| Sector Tank   | Excitation: Harmonic Pitching   |
|---|---|
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ .  |   |
| 1. Boundary conditions:   |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = -i\omega z \theta_0 e^{i\omega t} \cos \phi$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = i\omega r \theta_0 e^{i\omega t} \cos \phi$ |
| (c) $\left( \frac{\partial^2 \phi}{\partial z^2} + g \frac{\partial \phi}{\partial z} \right)_{z=+h/2} = 0$   | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = 0$                           |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\bar{\alpha}} = i\omega z \theta_0 e^{i\omega t} \sin \bar{\alpha}$   |   |
| 2. Velocity potential:  |   |
| $\phi = -i\omega \theta_0 e^{i\omega t} \left[ rz \cos \phi - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_{m/2\alpha}(\rho_{mn}) \cos \bar{\phi}_m (A_{mn} \cosh \zeta_{mn} + B_{mn} \sinh \zeta_{mn}) \right]$ |   |
| 3. Natural angular frequency:   |   |
| $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$   |   |
| where $\epsilon_{mn}$ are zeros of $J'_{m/2\alpha}(\epsilon_{mn}) = 0$ .  |   |

Table 4-15. Liquid Force and Moment Resultants

| Sector Tank  | Excitation: Harmonic Pitching  |
|--|--|
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ and pitching about the x-axis, $\chi = \chi_0 e^{i\omega t}$ .  |  |
| 1. $F_x = -m_L g \begin{pmatrix} \theta \\ 0 \end{pmatrix} e^{i\omega t} + m_L \omega^2 \begin{pmatrix} \theta \\ 0 \end{pmatrix} \chi_0 e^{i\omega t}$  | $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ 4(-1)^{m+1} \left( \frac{A_{mn}}{C_{mn}} \right) \frac{\sin \bar{\alpha} \sinh(\kappa_{mn}/2)}{\bar{\alpha} a \kappa_{mn}} \right] \left[ \frac{\bar{\alpha}^2 J_{m/2\alpha}(\epsilon_{mn})}{\pi^2 2 - \bar{\alpha}^2} + L_0(\epsilon_{mn}) \right]$   |
| 2. $F_y = -m_L g \begin{pmatrix} 0 \\ \chi_0 \end{pmatrix} e^{i\omega t} - m_L \omega^2 \begin{pmatrix} 0 \\ \chi_0 \end{pmatrix} \theta_0 e^{i\omega t}$  | $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ 4 \left( \frac{A_{mn}}{C_{mn}} \right) \frac{[1 - (-1)^m \cos \bar{\alpha}] \sinh(\kappa_{mn}/2)}{\bar{\alpha} a \kappa_{mn}} \right] \left[ \frac{\bar{\alpha}^2 J_{m/2\alpha}(\epsilon_{mn})}{\pi^2 2 - \bar{\alpha}^2} + L_0(\epsilon_{mn}) \right]$  |
| 3. $M_y = -m_L g a \begin{pmatrix} \theta \\ 0 \end{pmatrix} \chi_0 e^{i\omega t} - m_L \omega^2 \begin{pmatrix} \theta \\ 0 \end{pmatrix} \chi_0 e^{i\omega t} \left( \frac{1}{4(h/a)} \right) \left( 1 + \frac{\sin \bar{\alpha} \cos \bar{\alpha}}{\sin^2 \bar{\alpha}} \right) + m_L a^2 \omega^2 \begin{pmatrix} \theta \\ 0 \end{pmatrix} \chi_0 e^{i\omega t} \left\{ \frac{1}{8} \left( 1 + \frac{\sin \bar{\alpha} \cos \bar{\alpha}}{\sin^2 \bar{\alpha}} \right) - 2(h/a)^2/3 \right\}$ | $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{2(-1)^{m+1} \sin \bar{\alpha}}{\bar{\alpha} a^2 \epsilon_{mn}} \left[ \left( \frac{B_{mn}}{D_{mn}} \right) \left[ \cosh \left( \frac{\kappa_{mn}}{2} \right) - \frac{2}{\kappa_{mn}} \sinh \left( \frac{\kappa_{mn}}{2} \right) \right] \left[ \frac{\bar{\alpha}^2 J_{m/2\alpha}(\epsilon_{mn})}{\pi^2 2 - \bar{\alpha}^2} + L_0(\epsilon_{mn}) \right] \right]$ |
| $+ \frac{\bar{\alpha}^2 2 L_2(\epsilon_{mn})}{(\pi^2 2 - \bar{\alpha}^2) \kappa_{mn}} \left[ \left( \frac{A_{mn}}{C_{mn}} \right) \cosh \left( \frac{\kappa_{mn}}{2} \right) - \left( \frac{B_{mn}}{D_{mn}} \right) \sinh \left( \frac{\kappa_{mn}}{2} \right) \right] \left\{ \frac{2mga \sin \bar{\alpha}}{3 \bar{\alpha}} \right\}$   |  |



Table 4-15. Liquid Force and Moment Resultants (continued)

| Sector Tank |  | Excitation: Harmonic Pitching |
|-------------|--|-------------------------------|
| 4.          | $M_x = m_L g a \left( \frac{\theta_0}{\chi_0} \right) e^{i\omega t} \left( \frac{1}{4(h/a)} \right) \left( \frac{\sin^2 \bar{\alpha}}{1 - \sin \bar{\alpha} \cos \bar{\alpha}} \right) - m_L a^2 \omega^2 \left( \frac{\theta_0}{\chi_0} \right) e^{i\omega t} \left\{ \frac{1}{8} \left( \frac{\sin^2 \bar{\alpha}}{1 - \sin \bar{\alpha} \cos \bar{\alpha}} \right) \right.$ $- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{2 \left[ 1 - (-1)^m \cos \bar{\alpha} \right]}{\bar{\alpha}^2 \epsilon_{mn}} \left[ \left( \frac{B_{mn}}{D_{mn}} \right) \left[ \cosh \left( \frac{\kappa_{mn}}{2} \right) - \frac{2}{\kappa_{mn}} \sinh \left( \frac{\kappa_{mn}}{2} \right) \right] \left[ \frac{\bar{\alpha}^2 J_m / 2n (\epsilon_{mn})}{\pi^2 m^2 - \bar{\alpha}^2} + L_o (\epsilon_{mn}) \right] \right.$ $+ \frac{\bar{\alpha}^2 \epsilon_{mn} L_2 (\epsilon_{mn})}{(\pi^2 m^2 - \bar{\alpha}^2) \kappa_{mn}} \left[ \left( \frac{A_{mn}}{C_{mn}} \right) \cosh \left( \frac{\kappa_{mn}}{2} \right) - \left( \frac{B_{mn}}{D_{mn}} \right) \sinh \left( \frac{\kappa_{mn}}{2} \right) \right] \left. \right\} - \frac{2mga (1 - \cos \bar{\alpha})}{3\bar{\alpha}}$ |                               |

Table 4-16. Boundary Conditions, Velocity Potential, and Natural Frequency

| Sector Tank  | Excitation: Harmonic Roll  |
|--|--|
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .   |  |
| 1. Boundary condition:   |  |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = 0$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$   |
| (c) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0, \bar{\alpha}} = i\omega r \phi_0 e^{i\omega t}$   | (d) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$ |
| 2. Velocity potential:   |  |
| $\phi = i\omega a^2 \phi_0 e^{i\omega t} \left\{ \left( \frac{r}{a} \right)^2 \left( \phi - \frac{a}{2} \right) + \sum_{m=1}^{\infty} \frac{8\bar{\alpha}^2 \cos \bar{\phi}_m}{\pi(2m-1) \left[ (2m-1)^2 \pi^2 - 4\bar{\alpha}^2 \right]} \left[ \left( \frac{r}{a} \right)^{2m-1/2\alpha} - \left( \frac{r}{a} \right)^2 \frac{4\alpha}{2m-1} \right] \right.$ $+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{4\bar{\alpha} [4\alpha/(2m-1)] \epsilon_{2m-1,n} - e_{2m-1,n}}{\left[ (2m-1)^2 \pi^2 - 4\bar{\alpha}^2 \right] (\eta_{2m-1,n}^2 - 1) \cosh \kappa_{2m-1,n}} \cos \bar{\phi}_m \cosh (\kappa_{2m-1,n} + \zeta_{2m-1,n}) \left. \right\}$ |  |
| 3. Natural angular frequency:  |  |
| $\omega_{2m-1,n}^2 = \frac{g}{a} e_{2m-1,n} \tanh \kappa_{2m-1,n}$ <p>where <math>\epsilon_{2m-1,n}</math> are solutions of <math>J'_{4m-2}(\epsilon_{2m-1,n}) = 0</math>.</p>   |  |

Table 4-17. Liquid Force and Moment Resultants

| Sector Tank  | Excitation: Harmonic Roll |
|--|---------------------------|
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .   |                           |
| 1. $\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m_L a \omega^2 \phi_0 e^{i\omega t} \left\{ 2 \left( \frac{\cos \bar{\alpha} - 1}{\sin \bar{\alpha}} + \frac{\sin \bar{\alpha}}{3} \right) - \frac{16 \bar{\alpha} \sin \bar{\alpha}}{\pi} \sum_{m=1}^{\infty} \left[ \frac{2\alpha / (2m + 2\alpha - 1) - 4\alpha / 3(2m - 1)}{(2m - 1) [(2m - 1)^2 \pi^2 - 4 \bar{\alpha}^2]} \right] \right.$  |                           |
| $\left. + \frac{\bar{\alpha}^2}{(2m - 1)^2 [(2m - 1)^2 \pi^2 - \bar{\alpha}^2]} [(2m - 1) + 4\alpha] \right\} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{8 [(4\alpha / 2m - 1) f_{2m-1,n} - e_{2m-1,n}] \sin \bar{\alpha} \tanh \kappa_{2m-1,n}}{(h/a) \epsilon_{2m-1,n} [(2m-1)^2 \pi^2 - 4 \bar{\alpha}^2] (n_{2m-1,n} - 1)} \left[ \frac{\bar{\alpha}^2 J_{(2m-1)/2\alpha}(\epsilon_{2m-1,n})}{(2m-1)^2 \pi^2 - \bar{\alpha}^2} + L_0(\epsilon_{2m-1,n}) \right] \right\}$ |                           |
| 2. $\begin{pmatrix} M_x \\ M_y \end{pmatrix} = m_L a^2 \omega^2 \phi_0 e^{i\omega t} \left\{ \left[ \frac{1}{5(h/a)} \left( \frac{1 + \cos \bar{\alpha} - 2 \sin \bar{\alpha} / \bar{\alpha}}{\sin \bar{\alpha} + 2 \cos \bar{\alpha} - 2/\bar{\alpha}} \right) + \frac{48 \bar{\alpha}^4 \sin \bar{\alpha}}{5 \pi^2 (h/a)} \sum_{m=1}^{\infty} \right. \right.$   |                           |
| $\left. \frac{2m-1 - 4\alpha}{(2m-1)^2 (2m-1 + 6\alpha) [(2m-1)^2 \pi^2 - 4 \bar{\alpha}^2]} \left[ \frac{4\alpha / (2m-1)}{(2m-1)^2 \pi^2 - 4 \bar{\alpha}^2} + \frac{4 \sin \bar{\alpha}}{a^2} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{1}{\cosh \kappa_{2m-1,n}} - 1 \right] \right] \right.$   |                           |
| $\left. \frac{[4\alpha / (2m-1)] f_{2m-1,n} - e_{2m-1,n}}{[(2m-1)^2 \pi^2 - 4 \bar{\alpha}^2] (n_{2m-1,n} - 1)} \left[ \tanh \left[ \frac{\kappa_{2m-1,n}}{2} \right] + \frac{2}{\kappa_{2m-1,n}} \left( \frac{1}{\cosh \kappa_{2m-1,n}} - 1 \right) \right] \right.$  |                           |
| $\left. \frac{\epsilon_{2m-1,n}}{[(2m-1)^2 \pi^2 - 4 \bar{\alpha}^2]} \left[ \frac{\bar{\alpha}^2 J_{2m-1/2\alpha}(\epsilon_{2m-1,n})}{(2m-1)^2 \pi^2 - \bar{\alpha}^2} + L_0(\epsilon_{2m-1,n}) \right] + \frac{2 \bar{\alpha}^2 \epsilon_{2m-1,n} L_2(\epsilon_{2m-1,n})}{[(2m-1)^2 \pi^2 - \bar{\alpha}^2] \kappa_{2m-1,n} \cosh \kappa_{2m-1,n}} \right\} + \frac{2mga}{3 \bar{\alpha}} \left( \frac{\cos \bar{\alpha}}{\sin \bar{\alpha}} - 1 \right)$                                |                           |

Table 4-17. Liquid Force and Moment Resultants (continued)

| Sector Tank   | Excitation: Harmonic Roll |
|---|---------------------------|
| $3. \quad M_z = m_L a^2 \omega^2 \phi_0 e^{i\omega t} \left\{ \frac{1}{2} - \frac{16 \alpha^{-2}}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2 [(2m-1)\pi + 2\alpha]^{-2}} \right. \\ \left. - \frac{16}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{[(4\alpha/2m-1)) f_{2m-1,n} - e_{2m-1,n}] L_1 (\epsilon_{2m-1,n}) \tanh \kappa_{2m-1,n}}{[(2m-1)^2 \pi^2 - 4\alpha^{-2}] (\eta_{2m-1,n}^2 - 1) \kappa_{2m-1,n}} \right\}$ |                           |

## IV. Rigid Tanks

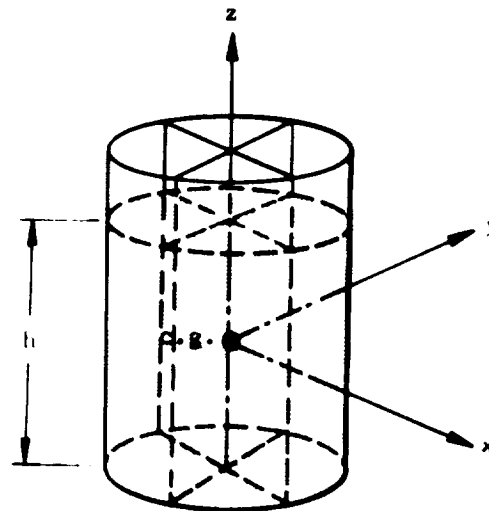
## 4.4 Quarter-Sectored Tank

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## IV. Rigid Tanks

## 4.4 Quarter-Sector Tank



Container: The tank is a right circular cylinder of radius  $a$  divided by longitudinal partitions intersecting at right angles,  $\alpha = \pi/2$ , and is filled with liquid to a depth  $h$ .

Coordinate System: The origin is located at the center of gravity of the undisturbed fluid and the  $x$ -axis must lie in a sector wall.

References: (4, 60)

Comments: The results given in this section are applicable to only one sector, i.e., the boundary conditions and resulting velocity potential, natural frequency, and force and moment resultants or responses due to liquid oscillations in only one sector.

Table 4-18. Boundary Conditions, Velocity Potential, and Natural Frequency

| Quarter-Sector Tank   |   | Excitation: Harmonic Translation |
|---|---|----------------------------------|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .   |   |                                  |
| 1. Boundary condition:  |   |                                  |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = i\omega x_0 e^{i\omega t} \cos \phi$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$                  |                                  |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$  | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = 0$ |                                  |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=a} = -i\omega x_0 e^{i\omega t} \sin a$   |   |                                  |
| 2. Velocity potential:  |   |                                  |
| $\phi = i\omega x_0 e^{i\omega t} \left[ r \cos \phi + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\bar{a}_m \bar{b}_n J_n(\rho_{mn}) \cos 2m\phi \cosh(\kappa_{mn} + \epsilon_{mn})}{(\eta_{mn}^2 - 1) \cosh \kappa_{mn}} \right]$ |   |                                  |
| 3. Natural angular frequency:   |   |                                  |
| $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$   |   |                                  |
| where $\epsilon_{mn}$ are roots of $J_{2m}'(\epsilon_{mn}) = 0$ . (see Table A-4).  |   |                                  |

Table 4-18. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

| Quarter-Sectored Tank  | Excitation: Harmonic Pitching   |
|--|---|
| Pitching about the y-axis $\theta = \theta_0 e^{i\omega t}$ .  |   |
| 1. Boundary conditions:  |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = -i\omega z \theta_0 e^{i\omega t} \cos \phi$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = i\omega r \theta_0 e^{i\omega t} \cos \phi$ |
| (c) $\left( \frac{\partial^2 \phi}{\partial z^2} + g \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = 0$  | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = 0$                           |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\alpha} = i\omega z \theta_0 e^{i\omega t} \sin \alpha$  |   |
| 2. Velocity potential:   |   |
| $\phi = -i\omega \theta_0 e^{i\omega t} \left[ rz \cos \phi - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\bar{A}_{mn} \cosh \zeta_{mn} + \bar{B}_{mn} \sinh \zeta_{mn}) J_{2m}(\rho_{mn}) \cos 2m\phi \right]$ |   |
| 3. Natural angular frequency:  |   |
| $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$  |   |
| where $\epsilon_{mn}$ are roots of $J'_{2m}(\epsilon_{mn}) = 0$ . (see Table A-4).   |   |



Table 4-19. Liquid Force and Moment Resultants

| Quarter-Sectored Tank   | Harmonic Translation and/or Pitching |
|---|--------------------------------------|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .   |                                      |
| 1. $\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m_L \omega^2 x_0 e^{i\omega t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \begin{pmatrix} (-1)^{m+1} \\ 1 \end{pmatrix} \frac{4 \bar{a}_m \bar{b}_{mn} \tanh \kappa_{mn}}{\pi a_{mn} (\eta_{mn}^2 - 1) \kappa_{mn}} \left( \frac{J_{2m}(\epsilon_{mn})}{4 m^2 - 1} + L_0(\epsilon_{mn}) \right) \right]$   |                                      |
| 2. $\begin{pmatrix} M_x \\ M_y \end{pmatrix} = \mp m_L a \omega^2 x_0 e^{i\omega t} \left[ \left( \frac{1/\pi}{1/4} \right) \frac{1}{(h/a)} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{1}{(-1)^{m+1}} \right) \frac{2 \bar{a}_m \bar{b}_{mn}}{\pi a_{mn} (\eta_{mn}^2 - 1) \epsilon_{mn}} \left\{ \left[ \frac{J_{2m}(\epsilon_{mn})}{4 m^2 - 1} + L_0(\epsilon_{mn}) \right] \right\} \right. \right.$<br>$\left. \left[ \tanh \kappa_{mn} + \frac{2}{\kappa_{mn}} \left( \frac{1}{\cosh \kappa_{mn}} - 1 \right) \right] + \frac{2 \epsilon_{mn}^2 L_2(\epsilon_{mn})}{(4m^2 - 1) \kappa_{mn} \cosh \kappa_{mn}} \right] \right] + \frac{4 m g a}{3 \pi}$   |                                      |
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ .  |                                      |
| 1. $\begin{pmatrix} F_x \\ F_y \end{pmatrix} = -m_L g \theta_0 e^{i\omega t} + \frac{8 m_L a \omega^2}{\pi} \theta_0 e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{(-1)^{m+1}}{1} \right) \frac{\bar{A}_{mn} \sinh(\kappa_{mn}/2)}{\kappa_{mn}} \left[ \frac{J_{2m}(\epsilon_{mn})}{4 m^2 - 1} + L_0(\epsilon_{mn}) \right]$   |                                      |
| 2. $\begin{pmatrix} M_x \\ M_y \end{pmatrix} = -\frac{m_L g a \theta_0 e^{i\omega t}}{2(h/a)} \begin{pmatrix} 1/\pi \\ 1/2 \end{pmatrix} + m_L a \omega^2 \theta_0 e^{i\omega t} \left\{ \left( \frac{1/4\pi}{1/8 - (h/a)^2/12} \right) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{4}{\pi \epsilon_{mn}} \left( \frac{1}{(-1)^{m+1}} \right) \left[ \bar{B}_{mn} \left( \cosh \frac{\kappa_{mn}}{2} \right. \right. \right.$<br>$\left. \left. - \frac{2}{\kappa_{mn}} \sinh \frac{\kappa_{mn}}{2} \right) \left( \frac{J_{2m}(\epsilon_{mn})}{4m^2 - 1} + L_0(\epsilon_{mn}) \right) + \frac{L_2(\epsilon_{mn}) \epsilon_{mn}^2}{(4m^2 - 1) \kappa_{mn}} - \left( \bar{A}_{mn} \cosh \frac{\kappa_{mn}}{2} - \bar{B}_{mn} \sinh \frac{\kappa_{mn}}{2} \right) \right] \right\} + \frac{4 m g a}{3 \pi}$ |                                      |



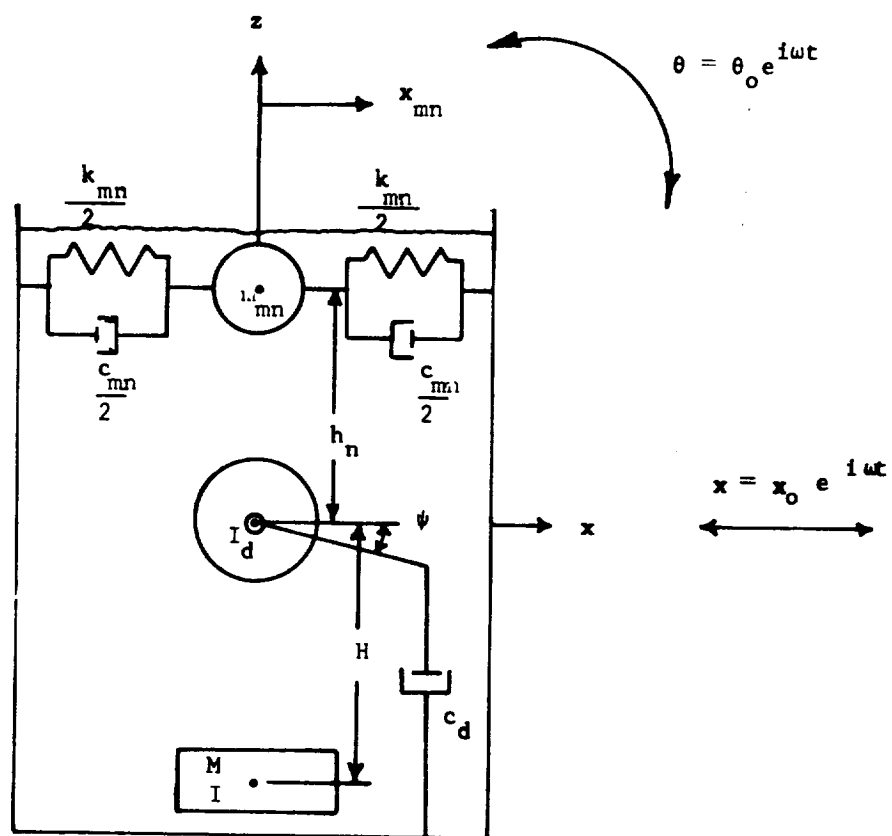
Table 4-20. Model Analysis

| Quarter-Sector Tank  | Spring-Mass Model |
|--|-------------------|
| Excitation: Harmonic Translation and/or Pitching   |                   |
| <p>Figure 4-19 shows a diagram of the spring-mass model used in representing the dynamic response of a liquid in a quarter-sectored tank when subjected to harmonic translation in the x-direction and/or pitching about the y-axis.</p> <p><u>Coordinate System:</u></p> <p>The origin is located at the center of gravity of the undisturbed liquid.</p> <p><u>Model Description:</u></p> <p>The components of the system are as follows:</p> <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the z-axis at a distance <math>H</math> below the coordinate origin.</li> <li>2. A set of movable masses <math>m_{mn}</math> is distributed along the z-axis when the tank is at rest at distances <math>h_{mn}</math> above the origin. These modal masses are constrained by spring-dashpot systems, having spring stiffness coefficients <math>k_{mn}</math> and viscous damping coefficients <math>c_{mn}</math>, to remain in the xy-plane and to move only in a direction parallel to the x-axis. Translational displacements of these masses with respect to the container are denoted by <math>x_{mn}</math>.</li> <li>3. A massless disc having a moment of inertia <math>I_d</math> is located at the coordinate origin. Its motion is confined to rotation about the y-axis and is cushioned by a dashpot having a viscous damping coefficient <math>c_d</math>. The angular displacement of the disc relative to the tank is defined by <math>\psi</math>.</li> </ol> <p><u>Equations of Motion:</u></p> <p>The equations, obtained through Lagrange's equations, are as follows:</p> <ol style="list-style-type: none"> <li>1. Force Equation: <math display="block">F_x = -M(\ddot{x} + H\ddot{\theta}) - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m_{mn} (\ddot{x} + \ddot{x}_{mn} + h_{mn} \ddot{\theta})</math> </li> <li>2. Moment Equation: <math display="block">M_y = -(I + MH^2)\ddot{\theta} - I_d(\ddot{\theta} + \ddot{\psi}) + g \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m_{mn} x_{mn} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m_{mn} h_{mn} (\ddot{x}_{mn} + h_{mn} \ddot{\theta})</math> </li> </ol> |                   |



Table 4-20. Model Analysis (continued)

| Quarter-Sector Tank   | Spring-Mass Model |
|---|-------------------|
| Excitation: Harmonic Translation and/or Pitching  |                   |
| <p data-bbox="342 472 813 510"><u>Equations of Motion (continued):</u></p> <p data-bbox="418 520 683 552">3. Disc Equation:</p> $I_d (\ddot{\theta} + \ddot{\psi}) + c_d \dot{\psi} = 0$ <p data-bbox="418 648 773 680">4. Slosh-Mass Equation:</p> $m_{mn} (\ddot{x} + \ddot{x}_{mn} + h_{mn} \ddot{\theta}) + m_{mn} \bar{g}_{mn} \omega_{mn} \dot{x}_{mn} + k_{mn} x_{mn} - m_{mn} g \theta = 0$ <p data-bbox="342 779 1485 863">From these equations, the model force in the x-direction and the moment about the y-axis can be found (see Table 4-21).</p> |                   |



Quarter-Sectored Tank

Spring-Mass Model

Excitation: Harmonic Translation and/or Pitching

Figure 4-19. Equivalent Mechanical Model

Table 4-21. Model Force and Moment Resultants

| Quarter-Sectored Tank   | Spring-Mass Model |
|---|-------------------|
| Excitation: Harmonic Translation and/or Pitching  |                   |
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .   |                   |
| 1. $F_x = m_L \omega^2 x_0 e^{i\omega t} \left[ 1 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m_{mn}}{m_L} \right) \left( \frac{1}{\eta_{mn}^2 - 1 + i \bar{g}_{mn} \eta_{mn}} \right) \right]$  |                   |
| 2. $M_y = m_L h \omega^2 x_0 e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m_{mn}}{m_L} \right) \left( \frac{h_{mn}}{h} + \frac{g}{h \omega^2} \right) \left( \frac{1}{\eta_{mn}^2 - 1 + i \bar{g}_{mn} \eta_{mn}} \right)$  |                   |
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ .  |                   |
| 1. $F_x = -m_L h \omega^2 \theta_0 e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m_{mn}}{m_L} \right) \left( \frac{h_{mn}}{h} + \frac{g}{h \omega^2} \right) \left( \frac{1}{\eta_{mn}^2 - 1 + i \bar{g}_{mn} \eta_{mn}} \right)$  |                   |
| 2. $M_y = -m_L h \omega^2 \theta_0 e^{i\omega t} \left[ \frac{I}{m_L h^2} + \frac{M h^2}{m_L h^2} + \frac{I_d}{m_L h^2} \left( 1 - \frac{\omega^2 I_d}{c_d^2 + \omega I_d^2} \right) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m_{mn} h^2}{m_L h^2} \right]$<br>$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m_{mn}}{m_L} \right) \left( \frac{h_{mn}}{h} + \frac{g}{h \omega^2} \right)^2 \left( \frac{1}{\eta_{mn}^2 - 1 + i \bar{g}_{mn} \eta_{mn}} \right) + i \omega \theta_0 e^{i\omega t} \left( \frac{c_d \omega I_d^2}{c_d^2 + \omega I_d^2} \right)$ |                   |





Table 4-22. Model Elements

| Quartered-Sector Tank                         |   | Spring-Mass Model |  |
|---|---|-------------------|--|
| Excitation: Harmonic Translation or Pitching  |   |                   |  |
| Natural Frequency                             | $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$   |                   |  |
| Spring Constant                               | $k_{mn} = m_{mn} \omega_{mn}^2$   |                   |  |
| Damping Coefficient of Slosh Mass             | $c_{mn} = m_{mn} \bar{g}_{mn} \omega_{mn}$  |                   |  |
| Ratio of Slosh Mass to Fluid Mass             | $\frac{m_{mn}}{m_L} = \frac{64 \tanh \kappa_{mn}}{\pi^2 \kappa_{mn} (\epsilon_{mn}^2 - 4m^2) J_{2m}^2(\epsilon_{mn})} \left[ \frac{\epsilon_{mn} J_{2m}(\epsilon_{mn})}{(4m^2 - 1)} \right. \quad (\text{fig. 4-20, 4-21})$ $\left. + 2 \sum_{\nu=0}^{\infty} J_{2m+2\nu+1}(\epsilon_{mn}) \right] \sum_{\nu=0}^{\infty} \frac{J_{2m+2\nu+1}(\epsilon_{mn})}{(2m+2\nu+3)(2m+2\nu-1)}$   |                   |  |
| Ratio of Fixed Mass to Fluid Mass             | $\frac{M}{m_L} = 1 - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{m_{mn}}{m_L}$  |                   |  |
| Ratio of Slosh Mass Coordinate to Fluid Depth | $\left  \frac{h_{mn}}{h} \right  = \frac{1}{2} \left[ 1 - \frac{4}{\kappa_{mn}} \tanh \frac{\kappa_{mn}}{2} \right] \quad (\text{fig. 4-22})$   |                   |  |
| Ratio of Fixed Mass Coordinate to Fluid Depth | $\left  \frac{H}{h} \right  = \frac{m_L}{M} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m_{mn}}{m_L} \right) \left( \frac{h_{mn}}{h} \right)$  |                   |  |
| Moment of Inertia of Solidified Fluid         | $\frac{I_s}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left[ \frac{1}{12} \left( \frac{h}{a} \right)^2 + \frac{1}{4} \right]$   |                   |  |
| Moment of Inertia of Disc ( $c_d = 0$ )       | $\frac{I_d}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left\{ \frac{1}{2} + 8 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} \bar{a}_m \bar{b}_n}{\pi a \epsilon_{mn}^2} \left[ \left( 1 - \frac{2}{\kappa_{mn}} \tanh \frac{\kappa_{mn}}{2} \right) \cdot \right. \right.$ $\left. \left( \frac{J_{2m}(\epsilon_{mn})}{4m^2 - 1} + L_0(\epsilon_{mn}) \right) - \frac{2\epsilon_{mn}^2 L_2(\epsilon_{mn}) \tanh \kappa_{mn}}{(4m^2 - 1) \kappa_{mn}} \right] \right\}$ |                   |  |
| Moment of Inertia of Fixed Mass               | $\frac{I}{m_L h^2} = \frac{I_s}{m_L h^2} - \frac{I_d}{m_L h^2} - \frac{M}{m_L} \left( \frac{H}{h} \right)^2 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m_{mn}}{m_L} \left( \frac{h_{mn}}{h} \right)^2$   |                   |  |



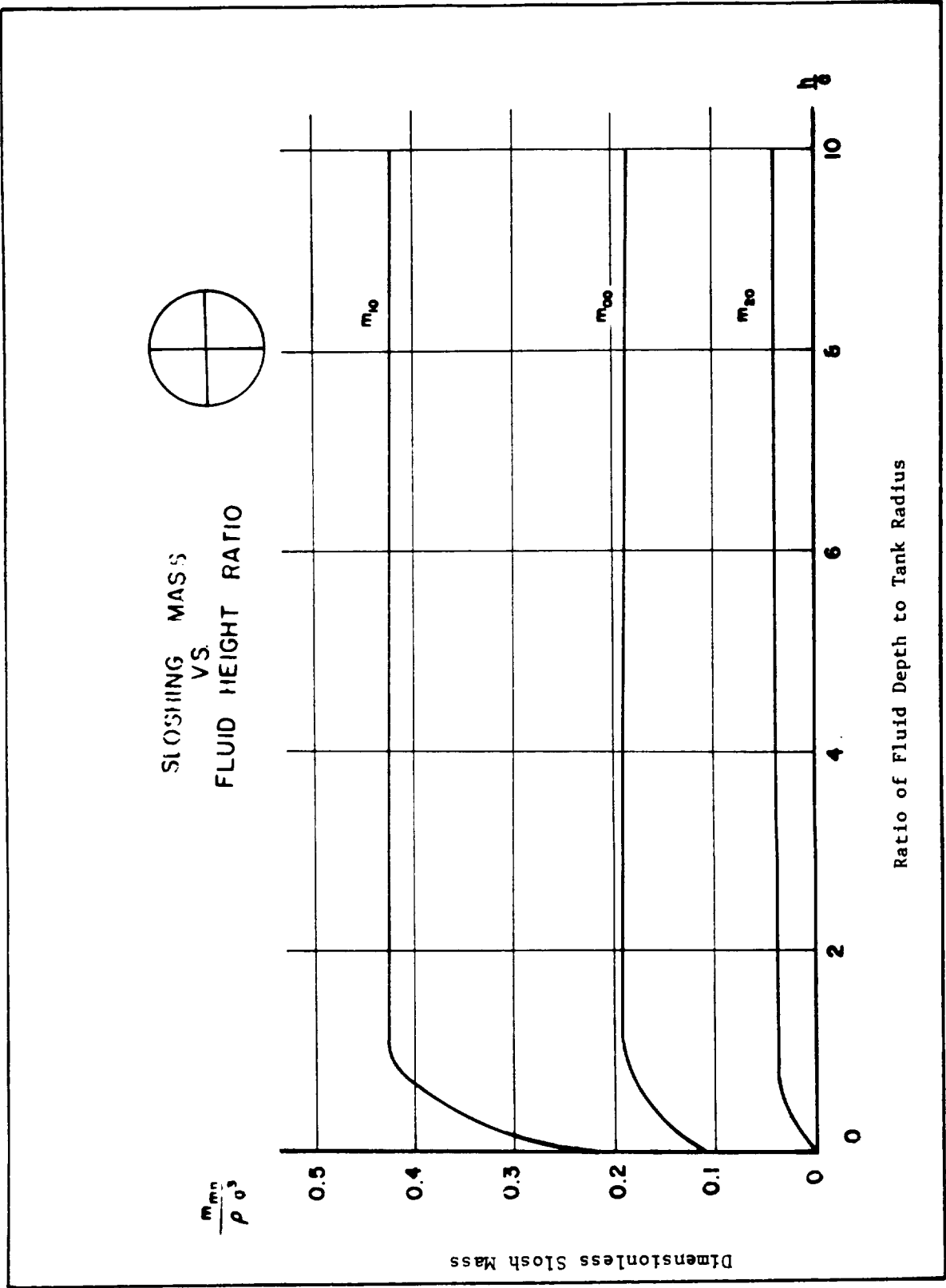


Figure 4-20. Model Element Graph

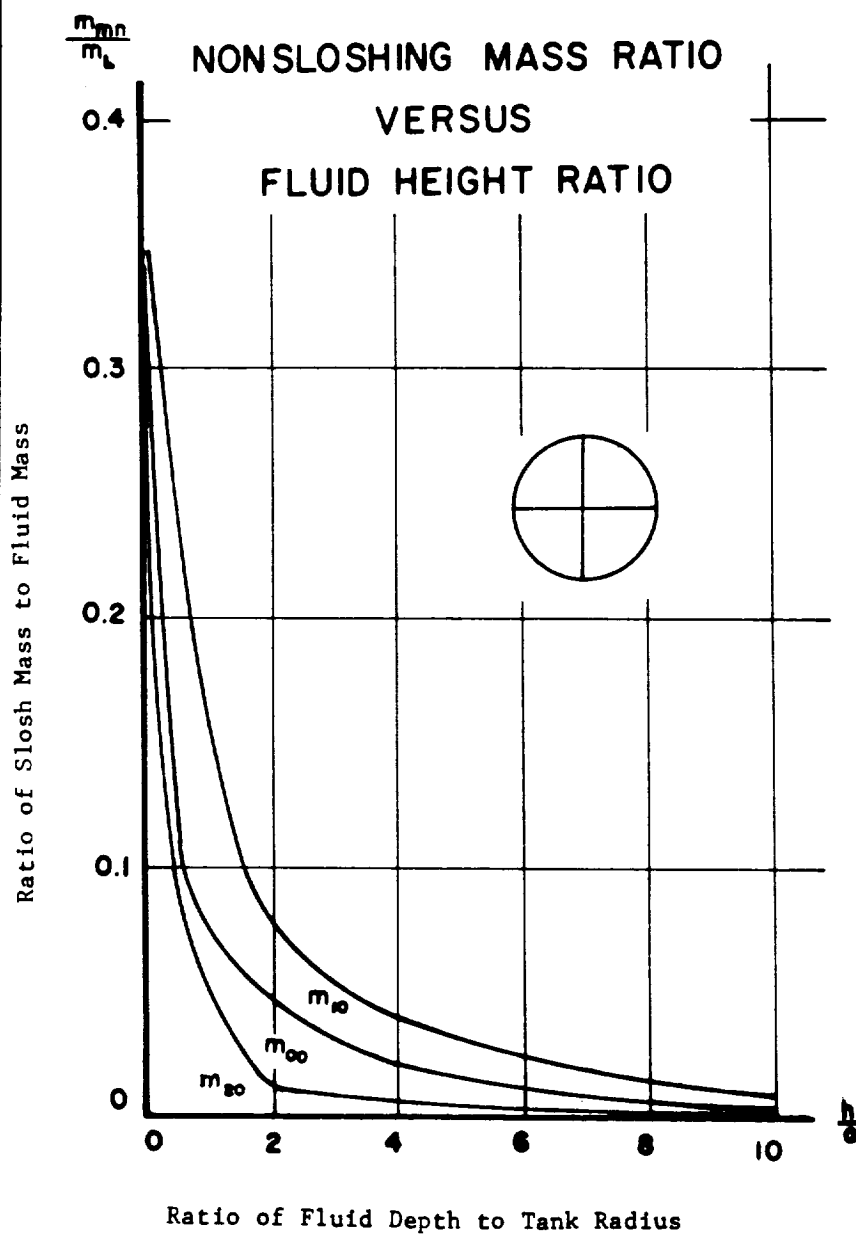


Figure 4-21. Model Element Graph

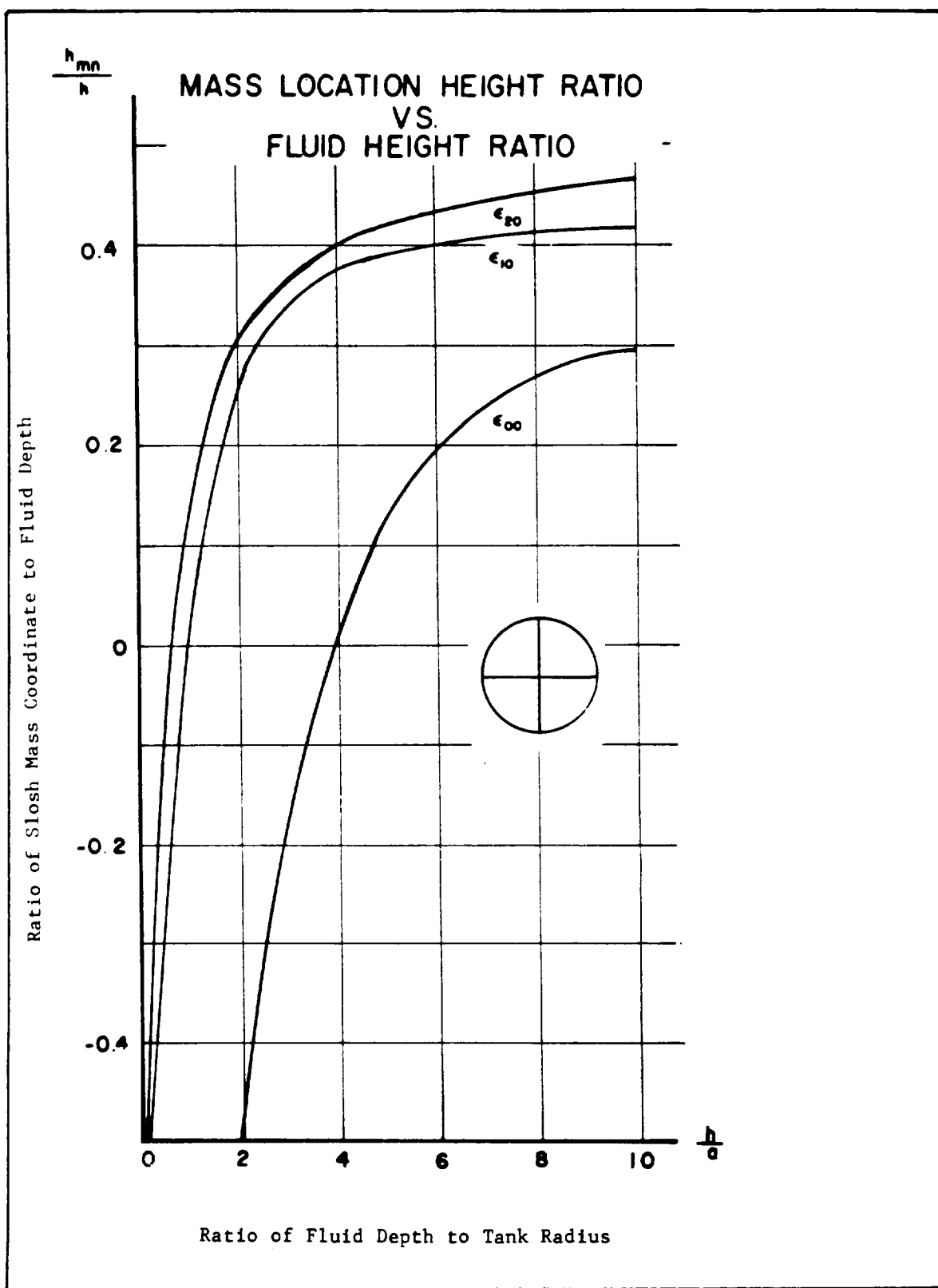


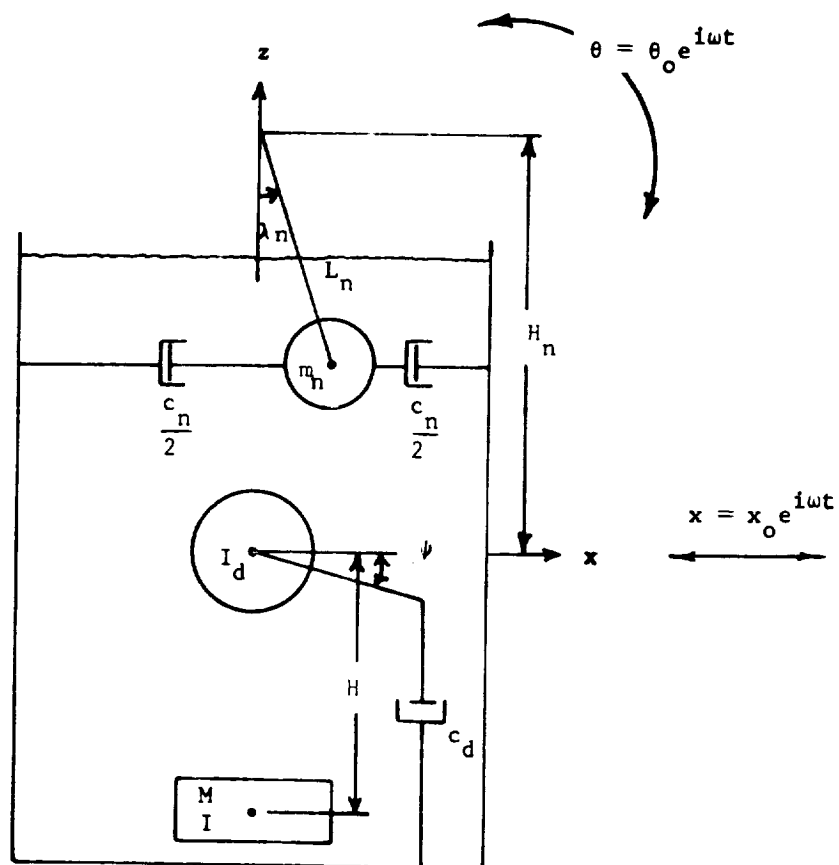
Figure 4-22. Model Element Graph

Table 4-23. Model Analysis

| Quarter-Sectored Tank  | Pendulum Model |
|--|----------------|
| Excitation: Harmonic Translation and/or Pitching   |                |
| <p>Figure 4-23 shows a diagram of the pendulum model used in representing the dynamic response of a liquid in a quarter-sectored tank when subjected to harmonic translation in the x-direction and/or pitching about the y-axis.</p>  |                |
| <p><u>Coordinate System:</u></p> <p>The origin is located at the center of gravity of the undisturbed liquid.</p>  |                |
| <p><u>Model Description:</u></p> <p>The components of the system are as follows:</p> <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the <math>z</math>-axis at a distance <math>H</math> below the coordinate origin.</li> <li>2. A set of movable masses <math>m_{mn}</math> is distributed along the <math>z</math>-axis when the tank is at rest. These modal masses are pendulums having massless lever arms of length <math>L_{mn}</math> attached to the <math>z</math>-axis at distances <math>H_{mn}</math> above the origin. They are constrained by dashpots having viscous damping coefficients <math>c_{mn}</math> to remain approximately in the <math>xy</math>-plane and to move approximately parallel to the <math>x</math>-axis. Angular displacements of the pendulum with respect to the tank (<math>z</math>-axis) are denoted by <math>\lambda_{mn}</math>.</li> <li>3. A massless disc having a moment of inertia <math>I_d</math> is located at the coordinate origin. Its motion is confined to rotation about the <math>y</math>-axis and is cushioned by a dashpot having a viscous damping coefficient <math>c_d</math>. The angular displacement of the disc relative to the tank is defined by <math>\psi</math>.</li> </ol> |                |
| <p><u>Equations of Motion:</u></p> <p>The equations, obtained through Lagrange's equations, are as follows:</p> <ol style="list-style-type: none"> <li>1. Force Equation: <math display="block">F_x = -M(\ddot{x} + H\ddot{\theta}) - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m_{mn} \left[ \ddot{x} + L_{mn} \ddot{\lambda}_{mn} + (H_{mn} - L_{mn}) \ddot{\theta} \right]</math> </li> <li>2. Moment Equation: <math display="block">M_y = - (I + MH^2) \ddot{\theta} - I_d (\ddot{\theta} + \ddot{\psi}) + g \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m_{mn} L_{mn} \lambda_{mn} \\ - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m_{mn} (H_{mn} - L_{mn}) \left[ L_{mn} \ddot{\lambda}_{mn} + (H_{mn} - L_{mn}) \ddot{\theta} \right]</math> </li> </ol>  |                |

Table 4-23. Model Analysis (continued)

| Quarter-Sector Tank   | Pendulum Model |
|---|----------------|
| Excitation: Harmonic Translation and/or Pitching  |                |
| <p data-bbox="375 457 844 485"><u>Equations of Motion (continued):</u></p> <p data-bbox="435 501 698 529">3. Disc Equation:</p> $I_d (\ddot{\theta} + \ddot{\psi}) + c_d \dot{\psi} = 0$ <p data-bbox="435 619 784 646">4. Slosh-Mass Equation:</p> $m_{mn} \left[ \ddot{x} + L_{mn} \ddot{\lambda}_{mn} + (H_{mn} - L_{mn}) \ddot{\theta} \right] + m_{mn} \bar{g}_{mn} \omega_{mn} L_{mn} \dot{\lambda}_{mn} - m_{mn} g \theta = 0$ <p data-bbox="375 783 1507 840">From these equations, the model force in the x-direction and the moment about the y-axis can be found (see Table 4-24).</p> |                |



Quarter-Sectored Tank

Pendulum Model

Excitation: Harmonic Translation and/or Pitching

Figure 4-23. Equivalent Mechanical Model



Table 4-24. Model Force and Moment Resultants

| Quarter-Sectored Tank   | Pendulum Model   |
|---|--|
| Excitation: Harmonic Translation and/or Pitching  |  |
| Translation in the x-direction, $x = x_o e^{i\omega t}$ .   |  |
| 1. $F_x = m_L \omega^2 x_o e^{i\omega t} \left[ 1 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m}{m_L} \right) \left( \frac{1}{\eta_{mn}^2 - 1 + i\tilde{g}_{mn} \eta_{mn}} \right) \right]$  |  |
| 2. $M_y = m_L h \omega^2 x_o e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m}{m_L} \right) \left( \frac{H_{mn} - L_{mn}}{h} + \frac{g}{h\omega^2} \right) \left( \frac{1}{\eta_{mn}^2 - 1 + i\tilde{g}_{mn} \eta_{mn}} \right)$  |  |
| Pitching about the y-axis, $\theta = \theta_o e^{i\omega t}$ .  |  |
| 1. $F_x = -m_L h \omega^2 \theta_o e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m}{m_L} \right) \left( \frac{H_{mn} - L_{mn}}{h} + \frac{g}{h\omega^2} \right) \left( \frac{1}{\eta_{mn}^2 - 1 + i\tilde{g}_{mn} \eta_{mn}} \right)$  |  |
| 2. $M_y = -m_L h \omega^2 \theta_o e^{i\omega t} \left[ \frac{I}{m_L h^2} + \frac{M h^2}{m_L h} + \frac{I_d}{m_L h} \left( 1 - \frac{\omega^2 I_d^2}{c_d^2 + \omega^2 I_d^2} \right) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m (H_{mn} - L_{mn})^2}{\eta_{mn}^2 - 1 + i\tilde{g}_{mn} \eta_{mn}} \right]$ |  |
|   | $+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m}{m_L} \right) \left( \frac{H_{mn} - L_{mn}}{h} + \frac{g}{h\omega^2} \right)^2 \left( \frac{1}{\eta_{mn}^2 - 1 + i\tilde{g}_{mn} \eta_{mn}} \right) + i\omega \theta_o e^{i\omega t} \left( \frac{c_d \omega^2 I_d^2}{c_d^2 + \omega^2 I_d^2} \right)$ |



Table 4-25. Model Elements.

| Quarter-Sector Tank                              | Pendulum Model   |
|--|--|
| Excitation: Harmonic Translation and/or Pitching |  |
| Natural Frequency                                | $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$  |
| Pendulum Length                                  | $L_{mn} = \frac{g}{\omega_{mn}^2}$   |
| Damping Coefficient of Slosh Mass                | $c_{mn} = m_{mn} \bar{g}_{mn} \omega_{mn}$   |
| Ratio of Slosh Mass to Fluid Mass                | $\frac{m_{mn}}{m_L} = \frac{64 \tanh \kappa_{mn}}{\pi^2 \kappa_{mn}^2 (\epsilon_{mn}^2 - 4m^2) J_{2m}^2(\epsilon_{mn})} \left[ \frac{\epsilon_{mn} J_{2m}(\epsilon_{mn})}{(4m^2 - 1)} \right.$ $\left. + 2 \sum_{\mu=0}^{\infty} J_{2m+2\mu+1}(\epsilon_{mn}) \right] \sum_{\mu=0}^{\infty} \frac{J_{2m+2\mu+1}(\epsilon_{mn})}{(2m+2\mu+3)(2m+2\mu-1)}$   |
| Ratio of Fixed Mass to Fluid Mass                | $\frac{M}{m_L} = 1 - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{m_{mn}}{m_L}$   |
| Ratio of Pendulum Mass Coordinate to Fluid Depth | $\frac{ H_{mn} - L_{mn} }{h} = \frac{1}{2} \left[ 1 - \frac{4}{\kappa_{mn}} \tanh \left( \frac{\kappa_{mn}}{2} \right) \right]$  |
| Ratio of Fixed Mass Coordinate to Fluid Depth    | $\frac{ H }{h} = \frac{m_L}{M} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{m_{mn}}{m_L} \right) \left( \frac{H_{mn} - L_{mn}}{h} \right)$   |
| Moment of Inertia of Solidified Fluid            | $\frac{I_s}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left[ \frac{1}{12} \left( \frac{h}{a} \right)^2 + \frac{1}{4} \right]$  |
| Moment of Inertia of Disc ( $c_d = 0$ )          | $\frac{I_d}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left\{ \frac{1}{2} + 8 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} a_m b_n}{\pi a \epsilon_{mn}^2} \left[ \left( 1 - \frac{2}{\kappa_{mn}} \tanh \frac{\kappa_{mn}}{2} \right) \right. \right.$ $\left. \left( \frac{J_{2m}(\epsilon_{mn})}{4m^2 - 1} + L_o(\epsilon_{mn}) \right) - \frac{2 \epsilon_{mn}^2 L_2(\epsilon_{mn}) \tanh \kappa_{mn}}{(4m^2 - 1) \kappa_{mn}} \right] \right\}$ |
| Moment of Inertia of Fixed Mass                  | $\frac{I}{m_L h^2} = \frac{I_s}{m_L h^2} - \frac{I_d}{m_L h^2} - \frac{m}{m_L} \left( \frac{H}{h} \right)^2 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m_{mn}}{m_L} \left[ \frac{H_{mn} - L_{mn}}{h} \right]^2$   |



Table 4-26. Boundary Conditions, Velocity Potential, and Natural Frequency

| Quarter-Sectored Tank   |  | Excitation: Harmonic Roll |
|---|--|---------------------------|
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .  |  |                           |
| 1. Boundary conditions:   |  |                           |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = 0$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$   |                           |
| (b) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0, \bar{a}} = i\omega r \phi_0 e^{i\omega t}$   | (d) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$ |                           |
| 2. Velocity potential:  |  |                           |
| $\phi = i\omega \phi_0 e^{i\omega t} \left\{ r^2 \left( \phi - \frac{\pi}{4} \right) + \sum_{m=1}^{\infty} \frac{2a^2 \cos(4m-2)\phi}{\pi(2m-1)[(2m-1)^2 - 1]} \left[ \left( \frac{r}{a} \right)^{4m-2} - \frac{(r/a)^2}{(2m-1)} \right] + \frac{2a^2}{\pi} \left( \frac{r}{a} \right)^2 \left[ \ln \left( \frac{r}{a} \right) + \frac{1}{2} \right] \cos 2\phi \right.$ $+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{2a^2}{\pi} \left[ \frac{[f_{2m-1,n} - e_{2m-1,n} (2m-1)] J_{4m-2}(\rho_{2m-1,n})}{(2m-1)[(2m-1)^2 - 1](\eta_{2m-1,n}^2 - 1) \cosh \kappa_{2m-1,n}} \right] \cosh(\kappa_{2m-1,n} + \epsilon_{2m-1,n}) \cos(4m-2)\phi$ $+ \sum_{n=0}^{\infty} \frac{a^2}{\pi} \left[ \frac{(2f_n - e_n) J_2(\rho_n) \cos 2\phi \cosh(\kappa_n + \epsilon_n)}{(\eta_n^2 - 1) \cosh \kappa_n} \right] \left. \right\}$ |  |                           |
| 3. Natural angular frequency:   |  |                           |
| $\omega_{2m-1,n}^2 = \frac{g}{a} \epsilon_{2m-1,n} \tanh \kappa_{2m-1,n} \quad \text{and} \quad \omega_n^2 = \frac{g}{a} \epsilon_n \tanh \kappa_n$   |  |                           |
| <p>where <math>\epsilon_{2m-1,n}</math> are solutions of <math>J'_{4m-2}(\epsilon_{2m-1,n}) = 0</math><br/> and <math>\epsilon_n</math> are roots of <math>J'_2(\epsilon_n) = 0</math>.</p>   |  |                           |

Table 4-27. Liquid Force and Moment Resultants

| Quarter-Sectored Tank   | Excitation: Harmonic Roll |
|---|---------------------------|
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .  |                           |
| $1. \quad F_x = -F_y = m_L \omega_a^2 \phi_0 e^{i\omega t} \left[ \frac{2}{3} - \frac{4}{\pi} + \frac{16}{9\pi^2} + \sum_{m=1}^{\infty} \left[ \frac{8 \left[ \frac{1}{(4m-1)} - \frac{1}{(3(2m-1))} \right]}{2(2m-1) \left[ (2m-1)^2 - 1 \right]} + \frac{1}{8m(2m-1)^2 \left[ (2m-1)^2 - 1/4 \right]} \right] \right. \\ \left. + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{4(2f_n - e_n) \tanh \kappa_n}{\pi^2 (n_n^2 - 1) \kappa_n} \left[ \frac{1}{3} J_2(\epsilon_n) + L_0(\epsilon_n) \right] + \frac{8 \left[ f_{2m-1,n} / (2m-1) - e_{2m-1,n} \right] \tanh \kappa_{2m-1,n}}{\pi^2 (h/a) (n_{2m-1,n}^2 - 1) \left[ (2m-1)^2 - 1 \right]} \epsilon_{2m-1,n} \right\} \right. \\ \left. \left[ \frac{J_{4m-2}(\epsilon_{2m-1,n})}{4 \left[ (2m-1)^2 - 1/4 \right]} + L_0(\epsilon_{2m-1,n}) \right] \right\} \right] $   |                           |
| $2. \quad M_x = M_y = m_L \omega_a^2 \phi_0 e^{i\omega t} \left[ \frac{1}{5(h/a)} \left( 1 - \frac{4}{\pi} \right) + \frac{4}{25 \pi^2 (h/a)} + \sum_{m=1}^{\infty} \frac{6(m-1)}{5 \pi^2 (h/a) \left[ (2m-1)^2 - 1/4 \right]} \left[ (2m-1)^2 - 1 \right] \left[ \frac{2m+1}{2} \right] \left[ (2m-1)^2 \right] \right. \\ + \sum_{n=0}^{\infty} \left\{ \frac{2(2f_n - e_n)}{\pi^2 (n_n^2 - 1) \epsilon_n^2} \left[ \tanh \kappa_n + \frac{2}{\kappa_n} \left( \frac{1}{\cosh \kappa_n} - 1 \right) \right] \left[ L_0(\epsilon_n) + \frac{1}{3} J_2(\epsilon_n) \right] + \frac{2 \epsilon_n^2 L_2(\epsilon_n)}{3 \kappa_n \cosh \kappa_n} \right\} \\ + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{4 \left[ f_{2m-1,n} / (2m-1) - e_{2m-1,n} \right]}{\pi^2 \left[ (2m-1)^2 - 1 \right] \epsilon_{2m-1,n}^2 (n_{2m-1,n}^2 - 1)} \left\{ \tanh \kappa_{2m-1,n} + \frac{2}{\kappa_{2m-1,n}} \left( \frac{1}{\cosh \kappa_{2m-1,n}} - 1 \right) \right\} \right. \\ \left. \left[ \frac{J_{4m-2}(\epsilon_{2m-1,n})}{4 \left[ (2m-1)^2 - 1/4 \right]} + L_0(\epsilon_{2m-1,n}) \right] + \frac{\epsilon_{2m-1,n}^2 L_2(\epsilon_{2m-1,n})}{2 \left[ (2m-1)^2 - 1/4 \right] \kappa_{2m-1,n} \cosh \kappa_{2m-1,n}} \right\} \right\} + \frac{4mga}{3\pi} $ |                           |

Table 4-27. Liquid Force and Moment Resultants (continued)

| Quarter-Sectored Tank   | Excitation: Harmonic Roll |
|---|---------------------------|
| $3. \quad M_z = m_L \omega^2 a^2 \phi_0 e^{i\omega t} \left\{ \frac{1}{2} - \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{1}{2^2 \pi^2 m^2 (2m-1)^2} + \frac{8(2f_n - e_n) L_1(\epsilon_n) \tanh \kappa_n}{\pi^2 (\eta_n^2 - 1) \kappa_n} \right] \right.$ $\left. + \frac{16 \left[ f_{2m-1,n} / (2m-1) - e_{2m-1,n} \right] L_1(\epsilon_{2m-1,n}) \tanh \kappa_{2m-1,n}}{\pi^2 \left[ (2m-1)^2 - 1 \right] (\eta_{2m-1,n}^2 - 1) \kappa_{2m-1,n}} \right\}$ |                           |





Table 4-28. Model Analysis

| Quarter-Sectored Tank  | Torsional Pendulum Model |
|--|--------------------------|
| Excitation: Harmonic Roll  |                          |
| <p>Figure 4-24 shows a diagram of the torsional pendulum model used in representing the dynamic response of a liquid in a quarter-sectored tank when subjected to harmonic roll about the z-axis.</p>  |                          |
| <p><u>Coordinate System:</u></p> <p>The origin is located at the center of gravity of the undisturbed liquid.</p>  |                          |
| <p><u>Model Description:</u></p> <ol style="list-style-type: none"> <li>1. A fixed mass having a mass moment of inertia <math>J</math> is rigidly connected to the tank and is located on the z-axis at the base of the tank.</li> <li>2. A disc having a mass moment of inertia <math>J_{mn}</math> is connected to a shaft having a torsional stiffness <math>K_{mn}</math> which is in turn rigidly attached to the tank. Viscous damping is introduced along the periphery of the disc, the damping coefficient of which is <math>c_{mn}</math>. Angular displacement of the disc relative to the tank is denoted by <math>\beta_{mn}</math>.</li> </ol> |                          |
| <p><u>Equations of Motion:</u></p> <p>The equations, obtained through Lagrange's equations, are as follows:</p> <ol style="list-style-type: none"> <li>1. Moment Equation: <math display="block">M_z = -J\ddot{\phi} + \sum_{m=1,3,5,\dots} \sum_{n=0}^{\infty} J_{mn} (\ddot{\phi} - \ddot{\beta}_{mn})</math> </li> <li>2. Slosh Mass Equation: <math display="block">\sum_{m=1,3,5,\dots} \sum_{n=0}^{\infty} J_{mn} (\ddot{\phi} + \ddot{\beta}_{mn}) + \sum_{m=1,3,5,\dots} \sum_{n=0}^{\infty} c_{mn} \dot{\beta}_{mn} + \sum_{m=1,3,5,\dots} \sum_{n=0}^{\infty} K_{mn} \beta_{mn} = 0</math> </li> </ol>   |                          |
| <p>From these equations, the model moment about the z-axis can be found (see Table 4-29).</p>  |                          |



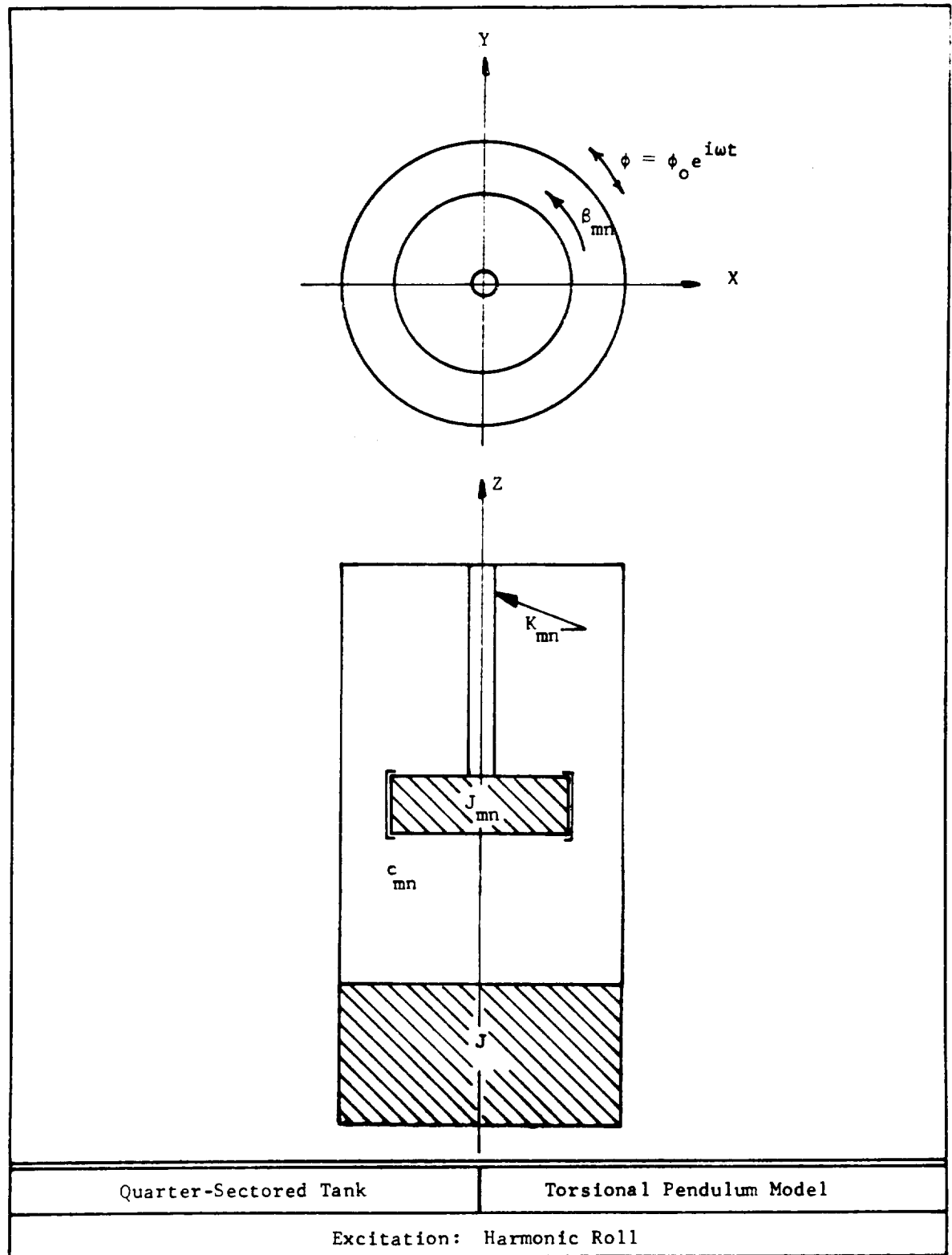


Figure 4-24. Equivalent Mechanical Model



Table 4-29. Moment Resultants

| Quarter-Sectored Tank  | Torsional Pendulum Model  |
|--|---------------------------|
|  | Excitation: Harmonic Roll |
|  |                           |
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .   |                           |
| $M_z = J\omega^2 \theta_0 e^{i\omega t} \left[ 1 + \sum_{m=1,3,5,\dots}^{\infty} n^2 \frac{J_{mn}}{J(\eta_{mn}^2 - 1 + i\bar{g}_{mn}\eta_{mn})} \right]$ |                           |



Table 4-30. Model Elements

| Quartered-Sector Tank                               | Torsional Pendulum Model   |
|---|--|
| Excitation: Harmonic Roll                           |  |
| Natural Frequency                                   | $\omega_{mn}^2 = \frac{g}{2} \epsilon_{mn} \tanh \kappa_{mn}$  |
| Torsional Stiffness of Shaft                        | $K_{mn} = J_{mn} \omega_{mn}^2$  |
| Damping Coefficient of Slosh Disc                   | $c_{mn} = J_{mn} \bar{g}_{mn} \omega_{mn}$   |
| Moment of Inertia of Slosh Disc ( $c_{mn} = 0$ )    | $J_{mn} = \frac{8m_L a^2}{\pi^2} \left[ - \frac{(2f_n - e_n) L_1(\epsilon_n) \tanh \kappa_n}{\kappa_n} + \right.$ $\left. - \frac{2(f_{mn} - m e_{mn}) L_1(\epsilon_{mn}) \tanh \kappa_{mn}}{m(m^2 - 1) \kappa_{mn}} \right]$  |
| Moment of Inertia of Slosh Disc ( $c_{mn} \neq 0$ ) | $J_{mn} = \frac{8m_L a^2 (\eta_{mn}^2 + i \bar{g}_{mn} \eta_{mn} - 1)}{\pi^2} \left[ - \frac{(2f_n - e_n) L_1(\epsilon_n) \tanh \kappa_n}{(\eta_n^2 - 1) \kappa_n} \right.$ $\left. - \frac{2(f_{mn} - m e_{mn}) L_1(\epsilon_{mn}) \tanh \kappa_{mn}}{m^2(m^2 - 1)(\eta_{mn}^2 - 1) \kappa_{mn}} \right]$ |
| Moment of Inertia of Solidified Fluid               | $J_s = \frac{1}{2} m_L a^2$  |
| Effective Moment of Inertia of Fluid                | $\bar{J} = J_s \left[ 1 - \frac{8}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2(m+1)^2} \right]$  |
| Moment of Inertia of Fixed Mass                     | $J = \bar{J} - \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=0}^{\infty} J_{mn}$   |

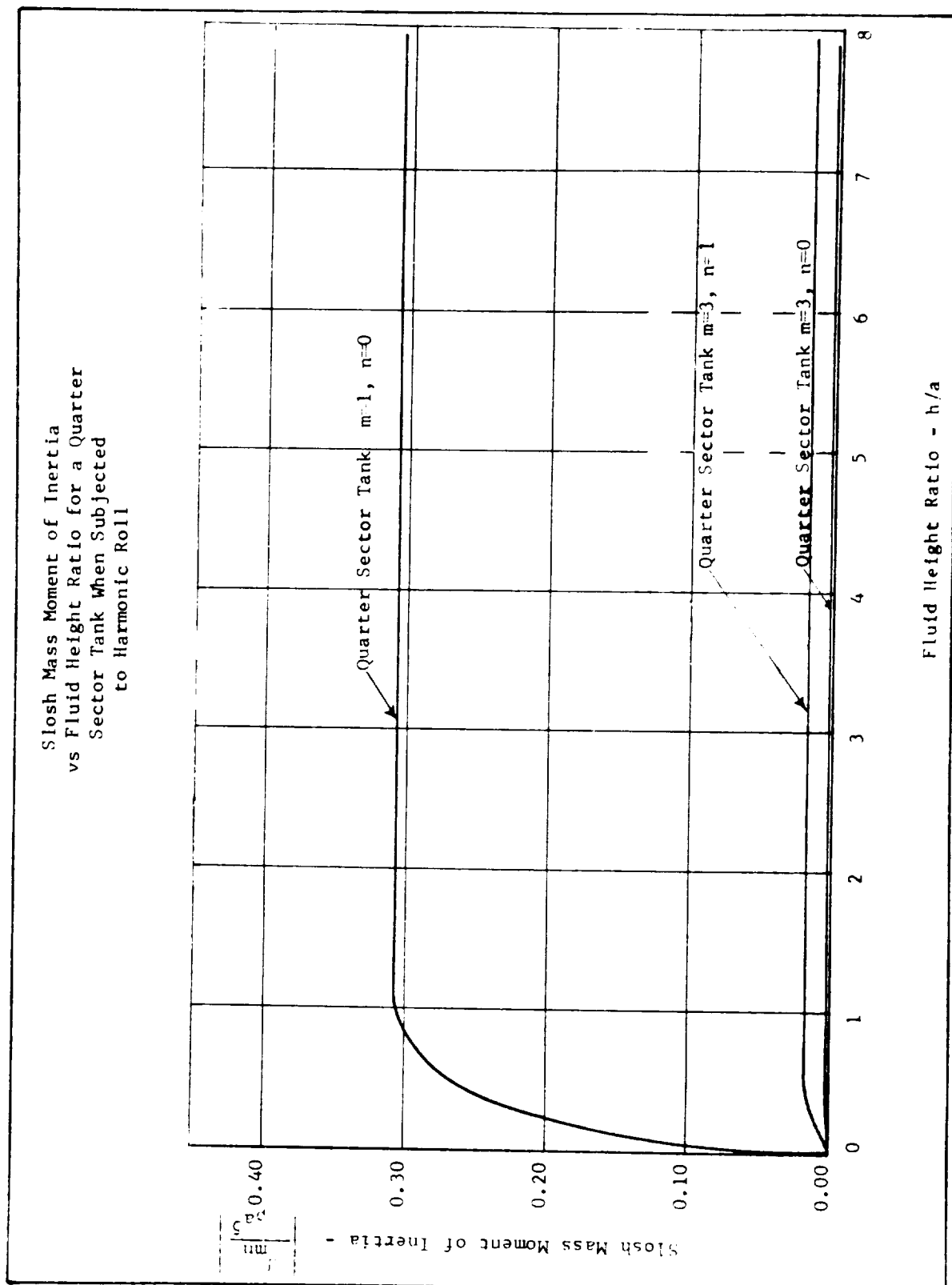


Figure 4-25. Model Element Graph



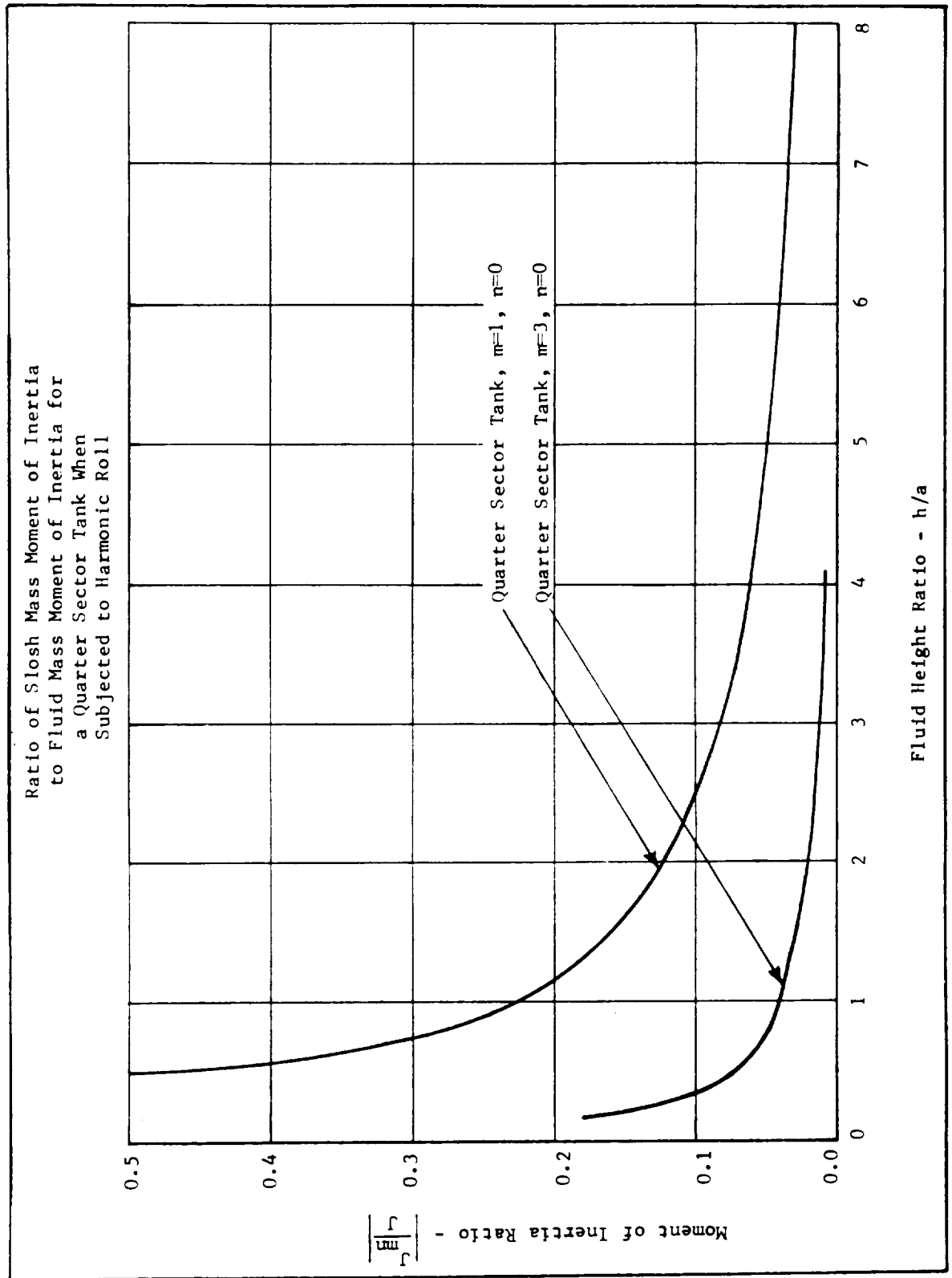


Figure 4-26. Model Element Tank



## IV. Rigid Tanks

## 4.5 Eighth-Sectored Tank

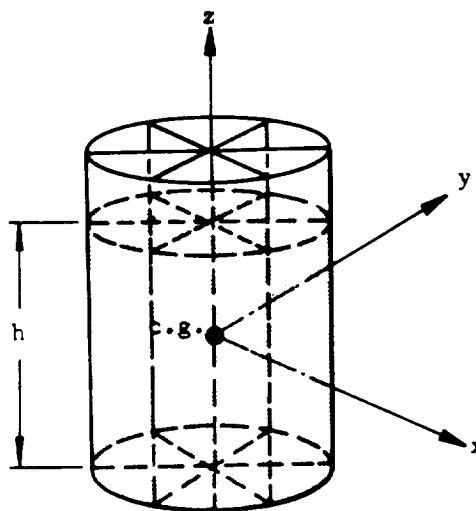
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## IV. Rigid Tanks

## 4.5 Eighth-Sector Tank



Container: The tank is a right circular cylinder of radius  $a$  divided by longitudinal partitions intersecting at equal angles,  $\alpha = \pi/4$ , and is filled with liquid to a depth  $h$ .

Coordinate System: The origin is located at the center of gravity of the undisturbed fluid and the x-axis must lie in a sector wall.

References: (60)

Comments: The results given in this section are applicable to only one sector, i.e., the boundary conditions and resulting velocity potential, natural frequency, and force and moment results are responses due to liquid oscillations in only one sector.



Table 4-31. Boundary Conditions, Velocity Potential, and Natural Frequency

| Eighth-Sector Tank  | Excitation: Harmonic Roll  |
|---|--|
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .  |  |
| 1. Boundary conditions:   |  |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a} = 0$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$   |
| (c) $\left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right)_{\phi=0, \bar{\alpha}} = i\omega r \phi_0 e^{i\omega t}$   | (d) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$ |
| 2. Velocity potential:  |  |
| $\phi = i\omega \phi_0 a e^{i\omega t} \left\{ \left( \frac{r}{a} \right)^2 \left( \phi - \frac{\pi}{8} \right) + \frac{2}{\pi} \sum_{m=1,3,5,\dots}^{\infty} \frac{\cos 4m\phi}{m(4m^2 - 1)} \left[ \left( \frac{r}{a} \right)^{4m} - \frac{1}{2m} \left( \frac{r}{a} \right)^2 \right] \right.$ $\left. + \frac{2}{\pi} \sum_{m=1,3,5,\dots}^{\infty} \frac{(\bar{r}_{mn} - 2m \bar{e}_{mn}) J_{4m}(\rho_{mn}) \cos 4m\phi \cosh(\kappa_{mn} + \tau_{mn})}{m(4m^2 - 1)(n_{mn}^2 - 1) \cosh \kappa_{mn}} \right\}$ |  |
| 3. Natural angular frequency:   |  |
| $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \rho_{mn}$   |  |
| where $\epsilon_{mn}$ are roots of $J'_{4m}(\epsilon_{mn}) = 0$ . (see table A-5)   |  |





Table 4-32. Liquid Moment Resultant

| Eighth-Sector Tank   | Excitation: Harmonic Roll |
|--|---------------------------|
| Roll about the z-axis, $\phi_0 e^{i\omega t}$ .  |                           |
| $M_z = m_L a^2 \omega^2 \phi_0 e^{i\omega t} \left[ \frac{1}{2} - \frac{4}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{2(2m+1)^2} - \frac{32}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \frac{(\bar{f}_{mn} - 2m \bar{e}_{mn}) L_2(\epsilon_{mn}) \tanh \kappa_{mn}}{m(4m^2 - 1)(\eta_{mn}^2 - 1) \kappa_{mn}} \right]$ |                           |



Table 4-33. Model Analysis

| Eighth-Sectored Tank  | Torsional Pendulum Model |
|---|--------------------------|
| Excitation: Harmonic Roll   |                          |
| <p data-bbox="342 447 1446 548">Figure 4-7 shows a diagram of the torsional pendulum model used in representing the dynamic response of a liquid in an eighth-sectored tank when subjected to harmonic roll about the z-axis.</p> <p data-bbox="342 604 607 632"><u>Coordinate System:</u></p> <p data-bbox="418 636 1485 678">The origin is located at the center of gravity of the undisturbed liquid.</p> <p data-bbox="342 735 607 762"><u>Model Description:</u></p> <ol data-bbox="418 766 1463 999" style="list-style-type: none"> <li>1. A fixed mass having a mass moment of inertia <math>J</math> is rigidly connected to the tank and is located on the z-axis at the base of the tank.</li> <li>2. A disc having a mass moment of inertia <math>J_{mn}</math> is connected to a shaft having a torsional stiffness <math>K_{mn}</math> which is in turn rigidly attached to the tank. Viscous damping is introduced along the periphery of the disc, the damping coefficient of which is <math>c_{mn}</math>. Angular displacement of the disc relative to the tank is denoted by <math>\beta_{mn}</math>.</li> </ol> <p data-bbox="342 1056 639 1083"><u>Equations of Motion:</u></p> <p data-bbox="418 1087 1419 1129">The equations, obtained through Lagrange's equation, are as follows:</p> <ol data-bbox="418 1140 711 1167" style="list-style-type: none"> <li>1. Moment Equation:</li> </ol> $M_z = -J\ddot{\beta}_{mn} - \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=0}^{\infty} J_{mn}(\ddot{\phi} + \ddot{\beta}_{mn})$ <ol data-bbox="418 1287 769 1314" style="list-style-type: none"> <li>2. Slosh Mass Equation:</li> </ol> $\sum_{m=1,3,5,\dots}^{\infty} \sum_{n=0}^{\infty} J_{mn}(\ddot{\phi} - \ddot{\beta}_{mn}) + \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=0}^{\infty} c_{mn}\dot{\beta}_{mn} - \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=0}^{\infty} K_{mn}\beta_{mn} = 0$ <p data-bbox="342 1570 1349 1633">From these equations, the model moment about the z-axis can be found (see Table 4-34).</p> |                          |



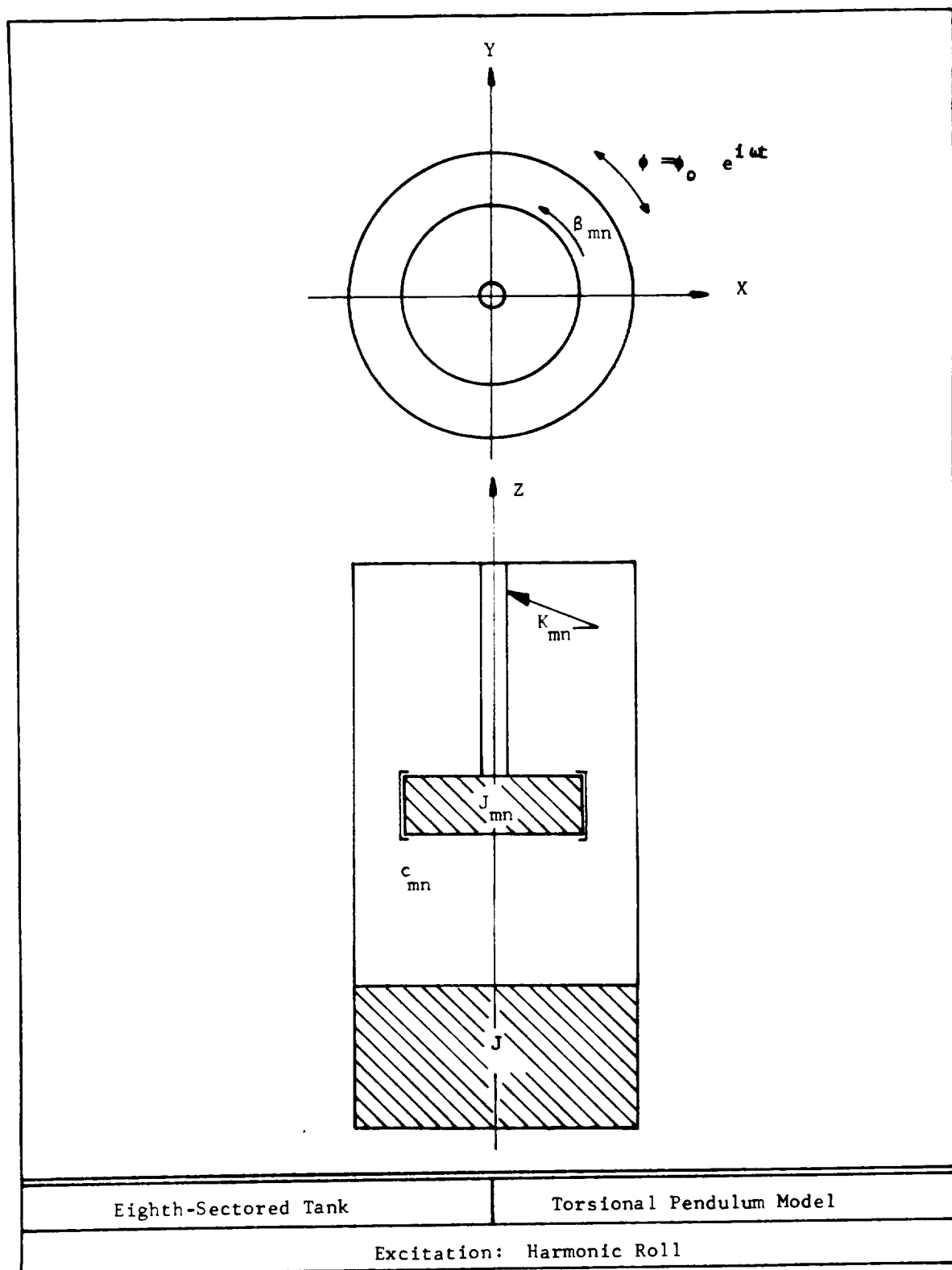


Figure 4-27. Equivalent Mechanical Model



Table 4-34. Model Moment Resultant

| Eighth-Sectored Tank   | Torsional Pendulum Model |
|--|--------------------------|
| Excitation: Harmonic Roll  |                          |
| <p data-bbox="516 1241 570 1734">Roll about the z-axis, <math>\phi = \phi_0 e^{i\omega t}</math>.</p> $M_z = J\omega^2 \phi_0 e^{i\omega t} \left[ 1 + \sum_{m=1,3,5,\dots}^{\infty} \frac{J_{mn}}{J(n_{mn}^2 - 1 + i g_{mn} n_{mn})} \right]$ |                          |





Table 4-35. Model Elements

| Eighth-Sectored Tank                                |  | Torsional Pendulum Model |  |
|---|--|--------------------------|--|
| Excitation: Harmonic Roll                           |  |                          |  |
| Natural Frequency                                   | $\omega_{mn}^2 = \frac{g}{a} \epsilon_{mn} \tanh \kappa_{mn}$  |                          |  |
| Torsional Stiffness of Shaft                        | $K_{mn} = J_{mn} \omega_{mn}^2$  |                          |  |
| Damping Coefficient of Slosh Disc                   | $c_{mn} = J_{mn} \bar{g}_{mn} \omega_{mn}$   |                          |  |
| Moment of Inertia of Slosh Disc ( $c_{mn} = 0$ )    | $J_{mn} = \frac{32m_L a^2}{\pi^2} \left[ \frac{-(\bar{f}_{mn} - 2m\bar{e}_{mn}) L_2(\epsilon_{mn}) \tanh \kappa_{mn}}{m(4m^2 - 1) \kappa_{mn}} \right]$ (fig. 4-28, 4-29)                                |                          |  |
| Moment of Inertia of Slosh Disc ( $c_{mn} \neq 0$ ) | $J_{mn} = \frac{-32m_L a^2 (\eta_{mn}^2 + i\bar{g}_{mn} \eta_{mn} - 1)}{\pi^2 (\eta_{mn}^2 - 1)} + \frac{(\bar{f}_{mn} - 2m\bar{e}_{mn}) L_2(\epsilon_{mn}) \tanh \kappa_{mn}}{m(4m^2 - 1) \kappa_{mn}}$ |                          |  |
| Moment of Inertia of Solidified Fluid               | $J_s = \frac{1}{2} m_L a^2$  |                          |  |
| Effective Moment of Inertia of Fluid                | $\bar{J} = J_s \left[ 1 - \frac{8}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2(m+1)^2} \right]$  |                          |  |
| Moment of Inertia of Fixed Mass                     | $J = \bar{J} - \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=0}^{\infty} J_{mn}$   |                          |  |



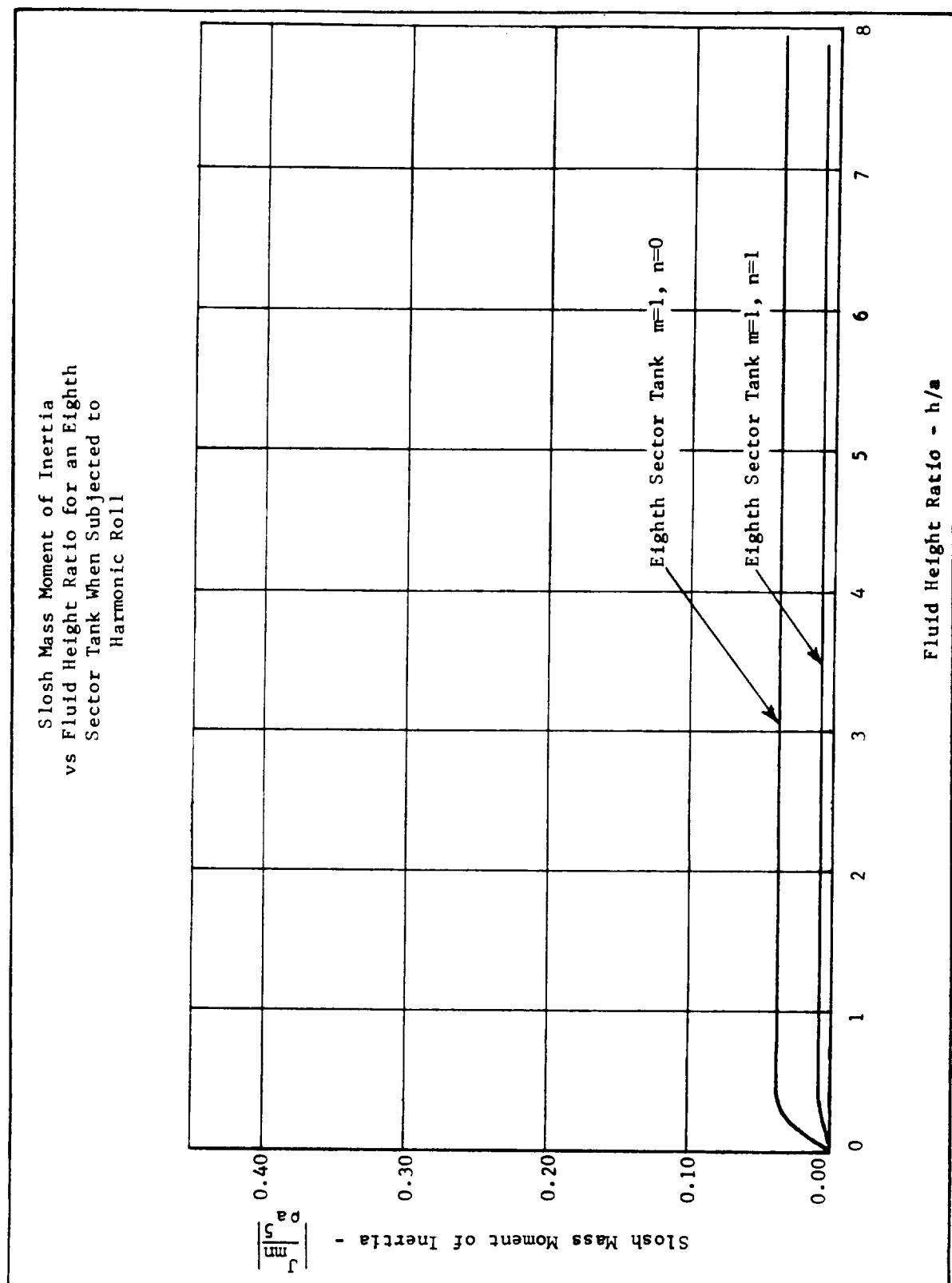


Figure 4-28. Model Element Graph



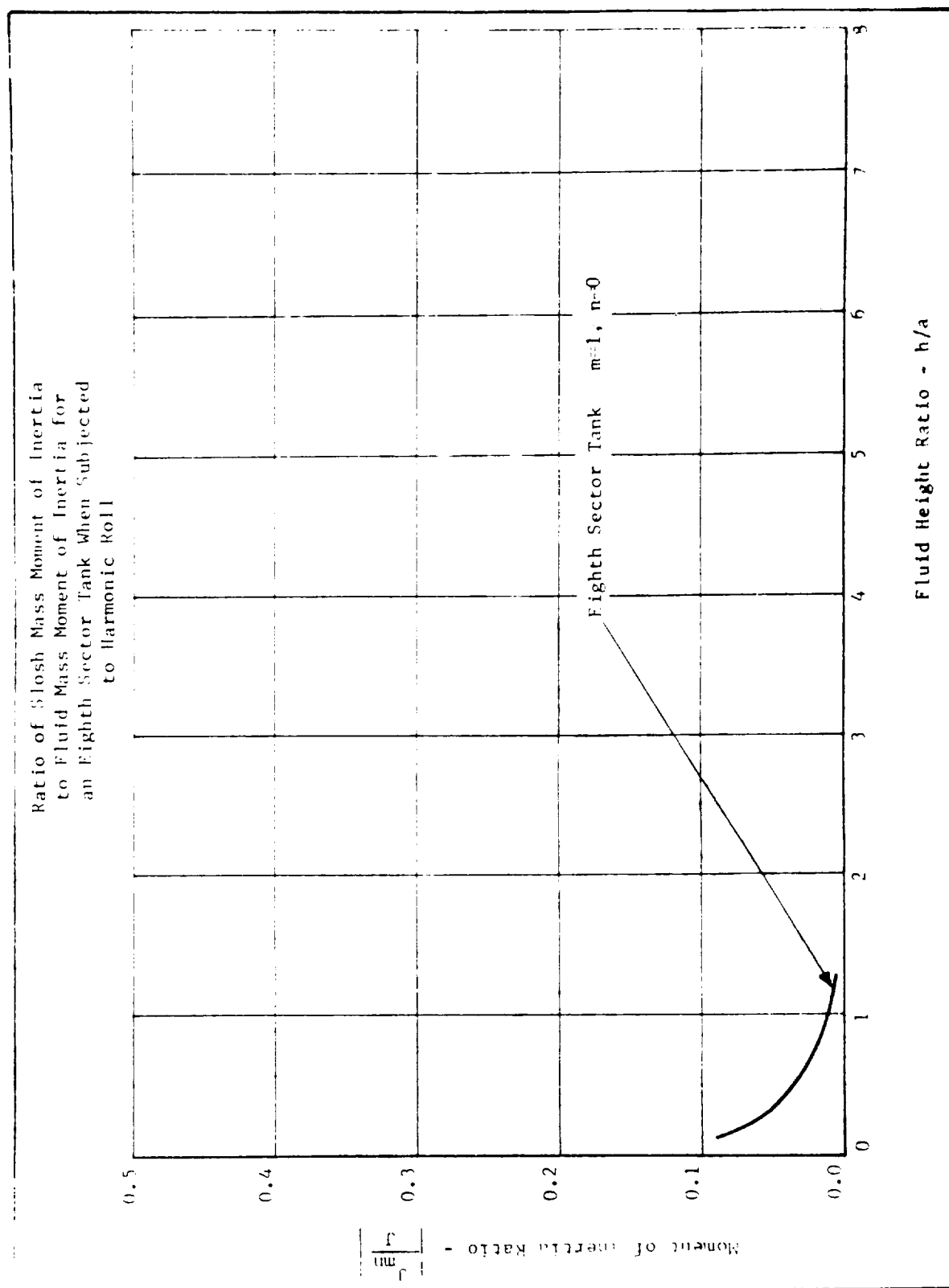


Figure 4-29. Model Element Graph



## IV. Rigid Tanks

## 4.6 Annular Tank

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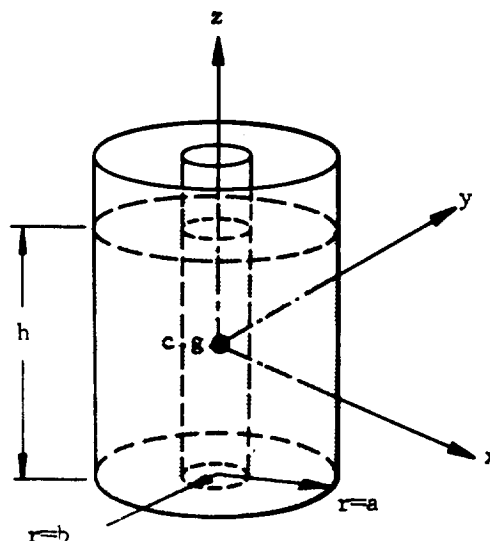
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## IV. Rigid Tanks

## 4.6 Annular Tank



Container: The tank consists of two concentric right circular cylinders having an inner radius  $r = b$  and an outer radius  $r = a$ . The space between the concentric walls is filled with liquid to a depth  $h$ .

Coordinate System: The origin is located at the center of gravity of the undisturbed fluid.

References: (52 and 63)

Comments: The terms  $\bar{I}$  and  $\bar{c}$ , occurring in the model elements, must be experimentally determined, generally from torsion spring experiments of a sealed container of fluid. Their values are dependent upon container shape and the type of liquid.



Table 4-36. Boundary Conditions, Velocity Potential, and Natural Frequency

| Annular Tank   | Excitation: Harmonic Translation   |
|--|--|
| <p>Translation in the x-direction, <math>x = x_0 e^{i\omega t}</math>.</p>   |  |
| <p>1. Boundary conditions:</p>   |  |
| <p>(a) <math>\left(\frac{\partial \phi}{\partial r}\right)_{r=a,b} = i\omega x_0 e^{i\omega t} \cos \phi</math></p>  | <p>(b) <math>\left(\frac{\partial \phi}{\partial z}\right)_{z=-h} = 0</math></p> |
| <p>(c) <math>\left(\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z}\right)_{z=0} = 0</math></p>   |  |
| <p>2. Velocity potential:</p>  |  |
| $\phi = i\omega x_0 e^{i\omega t} \cos \phi \left[ \frac{x}{a} + \sum_{n=0}^{\infty} \frac{A C_1(\rho_n) \cosh(\kappa_n + \zeta_n)}{(n_n^2 - 1) \cosh \kappa_n} \right]$ |  |
| <p>3. Natural angular frequency:</p>   |  |
| $\omega_n^2 = \frac{g}{a} \xi_n \tanh \kappa_n$  |  |
| <p>where <math>\xi_n</math> are roots of <math>\Delta_1(\xi_n) = J_1'(\xi_n) Y_1'(k\xi_n) - J_1'(k\xi_n) Y_1'(\xi_n) = 0</math>. (see Table A-6)</p>                     |  |



Table 4-36. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

| Annular Tank   | Excitation: Harmonic Pitching   |
|--|---|
| Pitching about the y-axis, $y = y_0 e^{i\omega t}$ .   |   |
| 1. Boundary conditions:  |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a,b} = -i\omega \theta_0 e^{i\omega t} z \cos \phi$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = i\omega \theta_0 e^{i\omega t} r \cos \phi$ |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=+h/2} = 0$  |   |
| 2. Velocity potential:   |   |
| $\phi = i\omega a^2 \theta_0 e^{i\omega t} \cos \phi \left\{ \frac{rz}{a^2} + \sum_{n=0}^{\infty} \frac{A_n C_1(\rho_n)}{(\eta_n^2 - 1) \cosh \kappa_n} \left[ \left( \frac{\gamma_n}{\xi_n} + \frac{h}{2a} \right) \cosh \left( \frac{\kappa_n}{2} + \zeta_n \right) - \frac{4\gamma_n}{\xi_n} \sinh \zeta_n \sinh \frac{\kappa_n}{2} - \frac{2}{\xi_n} \sinh \left( \frac{\kappa_n}{2} - \zeta_n \right) \right] \right\}$ |   |
| 3. Natural angular frequency:  |   |
| $\omega_n^2 = \frac{g}{a} \xi_n \tanh \kappa_n$  |   |
| where $\xi_n$ are roots of $\Delta_1(\xi_n) = J_1'(\xi_n) Y_1'(k\xi_n) - J_1'(k\xi_n) Y_1'(\xi_n) = 0$ . (see Table A-6)   |   |

Table 4-37. Liquid Force and Moment Resultants

| Annular Tank  | Excitation: Harmonic Translation |
|---|----------------------------------|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .   |                                  |
| 1. $F_x = m_L^2 \omega^2 x_0 e^{i\omega t} \left[ 1 + \sum_{n=0}^{\infty} \frac{A \left[ \frac{2}{\pi} \xi_n - k C_1(\sigma_n) \right] \tanh \kappa_n}{(1 - k^2)(\eta_n^2 - 1) \kappa_n} \right]$   |                                  |
| 2. $M_y = m_L a \omega^2 x_0 e^{i\omega t} \left\{ \frac{1 + k^2}{4(h/a)} + \sum_{n=0}^{\infty} \frac{A \left[ \frac{2}{\pi} \xi_n - k C_1(\sigma_n) \right]}{2(1 - k^2)(\eta_n^2 - 1)} \left[ \tanh \kappa_n + \frac{2}{\kappa_n} \left( \frac{2}{\cosh \kappa_n} - 1 \right) \right] \right\}$  |                                  |
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ .  |                                  |
| 1. $F_x = -m_L g \theta_0 e^{i\omega t} - m_L a \omega^2 \theta_0 e^{i\omega t} \sum_{n=0}^{\infty} \frac{A \left[ \frac{2}{\pi} \xi_n - k C_1(\sigma_n) \right]}{(1 - k^2)(\eta_n^2 - 1) \xi_n} \left[ \left( \frac{\gamma_n}{\kappa_n} + \frac{1}{2} \right) \tanh \kappa_n + \frac{2}{\kappa_n} \left( \frac{1}{\cosh \kappa_n} - 1 \right) \right]$ |                                  |
| 2. $M_y = -m_L g a \theta_0 e^{i\omega t} \frac{(1 + k^2)}{4(h/a)} - m_L^2 a \omega^2 \theta_0 e^{i\omega t} \left\{ \frac{1}{12} \left( \frac{h}{a} \right)^2 - \frac{1}{8} (1 + k^2) + \sum_{n=0}^{\infty} \frac{A \left[ \frac{2}{\pi} \xi_n - k C_1(\sigma_n) \right]}{(1 - k^2)(\eta_n^2 - 1) \xi_n} \right\}$                                     |                                  |
| $\left[ \left( \frac{h}{4a} - \frac{3\gamma_n}{2\xi_n} - \frac{4}{\kappa_n \xi_n} \tanh \kappa_n + \left( 2 - \frac{4\gamma_n}{\kappa_n} \right) \frac{1}{\xi_n \cosh \kappa_n} + \left( \frac{5\gamma_n}{\kappa_n \xi_n} + \frac{1}{2\xi_n} \right) \right] \right\}$  |                                  |

Table 4-38. Model Analysis

| Annular Tank   | Spring-Mass Model |
|--|-------------------|
| Excitation: Harmonic Translation and/or Pitching   |                   |
| <p>Figure 4-30 shows a diagram of the spring-mass model used in representing the dynamic response of a liquid in an annular tank when subjected to harmonic translation in the x-direction and/or pitching about the y-axis.</p>   |                   |
| <p><u>Coordinate System:</u></p>   |                   |
| <p>The origin is located at the center of gravity of the undisturbed liquid.</p>   |                   |
| <p><u>Model Description:</u></p>   |                   |
| <p>The components of the system are as follows:</p>  |                   |
| <p>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the <math>z</math>-axis at the distance <math>H</math> below the coordinate origin.</p>   |                   |
| <p>2. A set of movable masses <math>m_n</math> is distributed along the <math>z</math>-axis when the tank is at rest at distances <math>h_n</math> above the origin. These modal masses are constrained by spring-dashpot systems, having spring stiffness coefficients <math>k_n</math> and viscous damping coefficients <math>c_n</math>, to remain in the <math>xy</math>-plane and to move only in a direction parallel to the <math>x</math>-axis. Translational displacements of these masses with respect to the container are denoted by <math>x_n</math>.</p> |                   |
| <p>3. A massless disc having a moment of inertia <math>I_d</math> is located at the coordinate origin. Its motion is confined to rotation about the <math>y</math>-axis and is cushioned by a dashpot having a viscous damping coefficient <math>c_d</math>. The angular displacement of the disc relative to the tank is defined by <math>\psi</math>.</p>  |                   |
| <p><u>Equations of Motion:</u></p>   |                   |
| <p>The equations, obtained through Lagrange's equations, are as follows:</p>   |                   |
| <p>1. Force Equation:</p>  |                   |
| $F_x = -M(\ddot{x} + H\ddot{\theta}) - \sum_{n=1}^{\infty} m_n(\ddot{x} + \ddot{x}_n + h_n\ddot{\theta})$  |                   |
| <p>2. Moment Equation:</p>   |                   |
| $M_y = -(I + MH^2)\ddot{\theta} - I_d(\ddot{\theta} + \ddot{\psi}) + g \sum_{n=1}^{\infty} m_n x_n - \sum_{n=1}^{\infty} m_n h_n (\ddot{x}_n + h_n \ddot{\theta})$   |                   |

Table 4-38. Model Analysis (continued)

| Annular Tank   | Spring-Mass Model |
|--|-------------------|
| Excitation: Harmonic Translation and/or Pitching   |                   |
| <p data-bbox="228 422 706 453"><u>Equations of Motion (continued):</u></p> <p data-bbox="305 466 574 497">3.. Disc Equation:</p> $I_d (\ddot{\theta} + \ddot{\psi}) + c_d \dot{\psi} = 0 \quad 0$ <p data-bbox="305 579 662 611">4. Slosh-Mass Equation:</p> $\dot{m}_n (\ddot{x} + \ddot{x}_n + h_n \ddot{\theta}) + m_n \bar{g}_n \omega_n \dot{x}_n + k_n x_n - m_n g \theta = 0$ <p data-bbox="228 741 1369 800">From these equations, the model force in the x-direction and the moment about the y-axis can be found (see Table 4-39).</p> |                   |



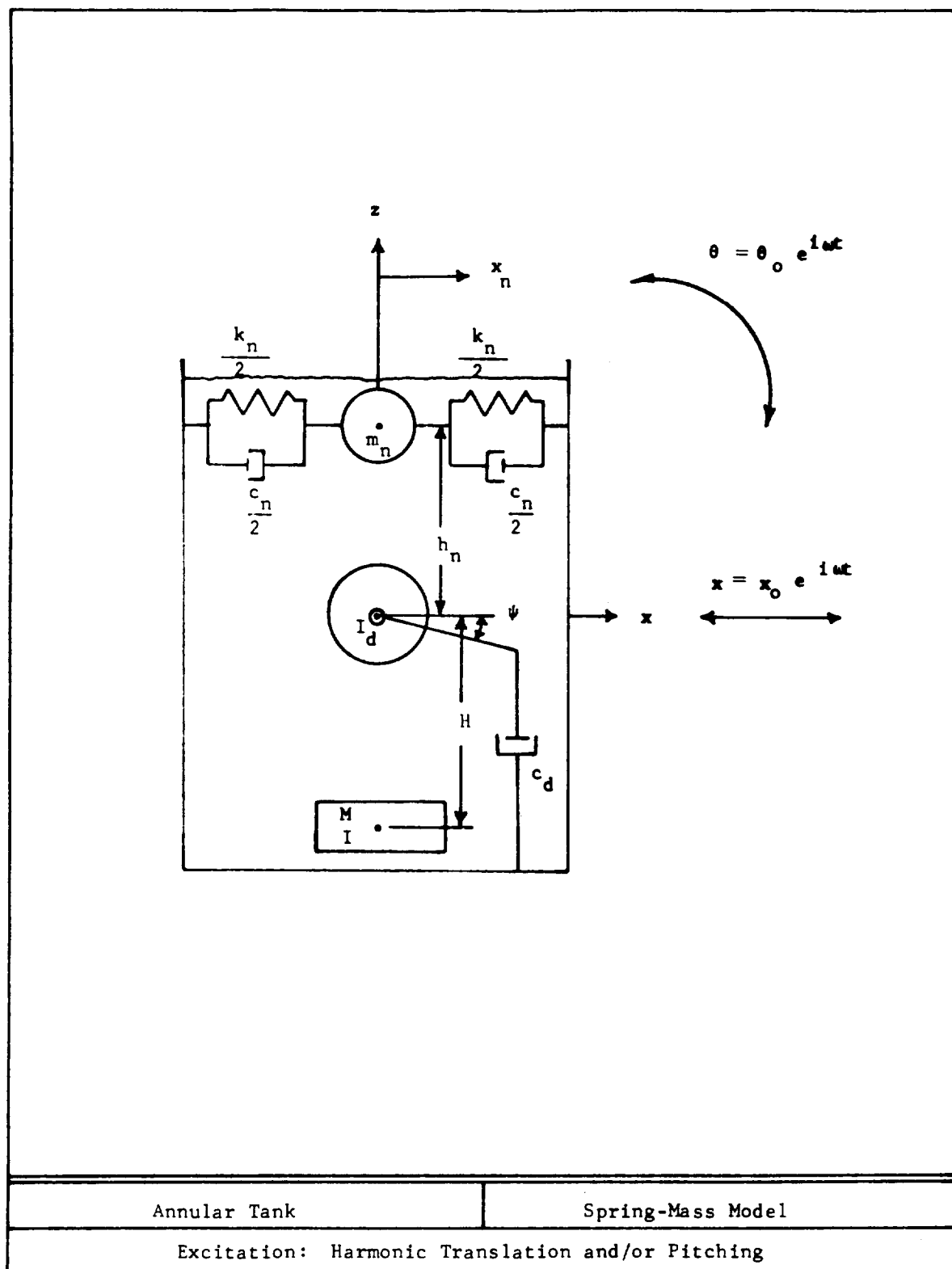


Figure 4-30. Equivalent Mechanical Model



Table 4-39. Model Force and Moment Resultants

| Annular Tank  | Spring-Mass Model  |
|---|--|
|   | Excitation: Harmonic Translation and/or Pitching   |
| Translation in the x-direction, $x = x_o e^{i\omega t}$ .   |  |
| 1. $F_x = m_L \omega^2 x_o e^{i\omega t} \left[ 1 + \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) \right]$   |  |
| 2. $M_y = m_L h \omega^2 x_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h}{h} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$   |  |
| Pitching about the y-axis, $\theta = \theta_o e^{i\omega t}$ .  |  |
| 1. $F_x = -m_L h \omega^2 \theta_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h}{h} + \frac{g}{h\omega^2} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$                               |  |
| 2. $M_y = -m_L h^2 \omega^2 \theta_o e^{i\omega t} \left[ \frac{1}{m_L h^2} + \frac{I_d}{m_L h^2} + \frac{I_d}{m_L h^2} \left( 1 - \frac{\omega^2 I_d}{c_d^2 + \omega^2 I_d^2} \right) + \sum_{n=0}^{\infty} \frac{m_n h^2}{m_L h^2} \right]$ |  |
|   | $+ \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h}{h} + \frac{g}{h\omega^2} \right)^2 \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) + i\omega \theta_o e^{i\omega t} \frac{c_d \omega^2 I_d}{c_d^2 + \omega^2 I_d^2}$ |



Table 4-40. Model Elements

| Annular Tank                                  |   | Spring-Mass Model |  |
|---|---|-------------------|--|
| Excitation: Harmonic Translation or Pitching  |   |                   |  |
| Natural Frequency                             | $\omega_n^2 = \frac{g}{a} \xi_n \tanh \kappa_n$   |                   |  |
| Spring Constant                               | $k_n = m_n \omega_n^2$ (fig. 4-31)  |                   |  |
| Damping Coefficient of Slosh Mass             | $c_n = m_n \bar{g}_n \omega_n$ (fig. 4-32)  |                   |  |
| Damping Coefficient of Disc                   | $c_d = \bar{c} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   |                   |  |
| Ratio of Slosh Mass to Fluid Mass             | $\frac{m_n}{m_L} = \frac{A_n \left[ \frac{2}{\pi} \xi_n - k C_1(\sigma_n) \right] \tanh \kappa_n}{(1-k^2) \kappa_n}$ (fig. 4-33)<br>(fig. 4-34)   |                   |  |
| Ratio of Fixed Mass to Fluid Mass             | $\frac{M}{m_L} = 1 - \sum_{n=0}^{\infty} \frac{m_n}{m_L}$ (fig. 4-35)   |                   |  |
| Ratio of Slosh Mass Coordinate to Fluid Depth | $\left  \frac{h_n}{h} \right  = \frac{1}{2} \left[ 1 - \frac{4}{\kappa_n} \tanh \frac{\kappa_n}{2} \right]$ (fig. 4-36)   |                   |  |
| Ratio of Fixed Mass Coordinate to Fluid Depth | $\left  \frac{H}{h} \right  = \frac{m_L}{M} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h_n}{h} \right)$ (fig. 4-37)  |                   |  |
| Moment of Inertia of Solidified Fluid         | $\frac{I_s}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left[ \frac{1}{12} \left( \frac{h}{a} \right)^2 + \frac{1}{4} \right]$   |                   |  |
| Moment of Inertia of Disc ( $c_d = 0$ )       | $\frac{I_d}{m_L h^2} = 4 \left( \frac{a}{h} \right)^2 \sum_{n=0}^{\infty} \frac{A_n \left[ \frac{2}{\pi} \xi_n - k C_1(\sigma_n) \right] \left[ 1 - 2/\kappa_n \tanh(\kappa_n/2) \right]}{(1-k^2) \xi_n}$ (fig. 4-38) |                   |  |
| Moment of Inertia of Disc ( $c_d \neq 0$ )    | $\frac{I_d}{m_L h^2} = \frac{I_s - \bar{I}}{m_L h^2} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   |                   |  |
| Moment of Inertia of Fixed Mass               | $\frac{I}{m_L h^2} = \frac{I_s}{m_L h^2} - \frac{I_d}{m_L h^2} - \frac{M}{m_L} \left( \frac{H}{h} \right) - \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{h_n}{h} \right)^2$ (fig. 4-39)                           |                   |  |



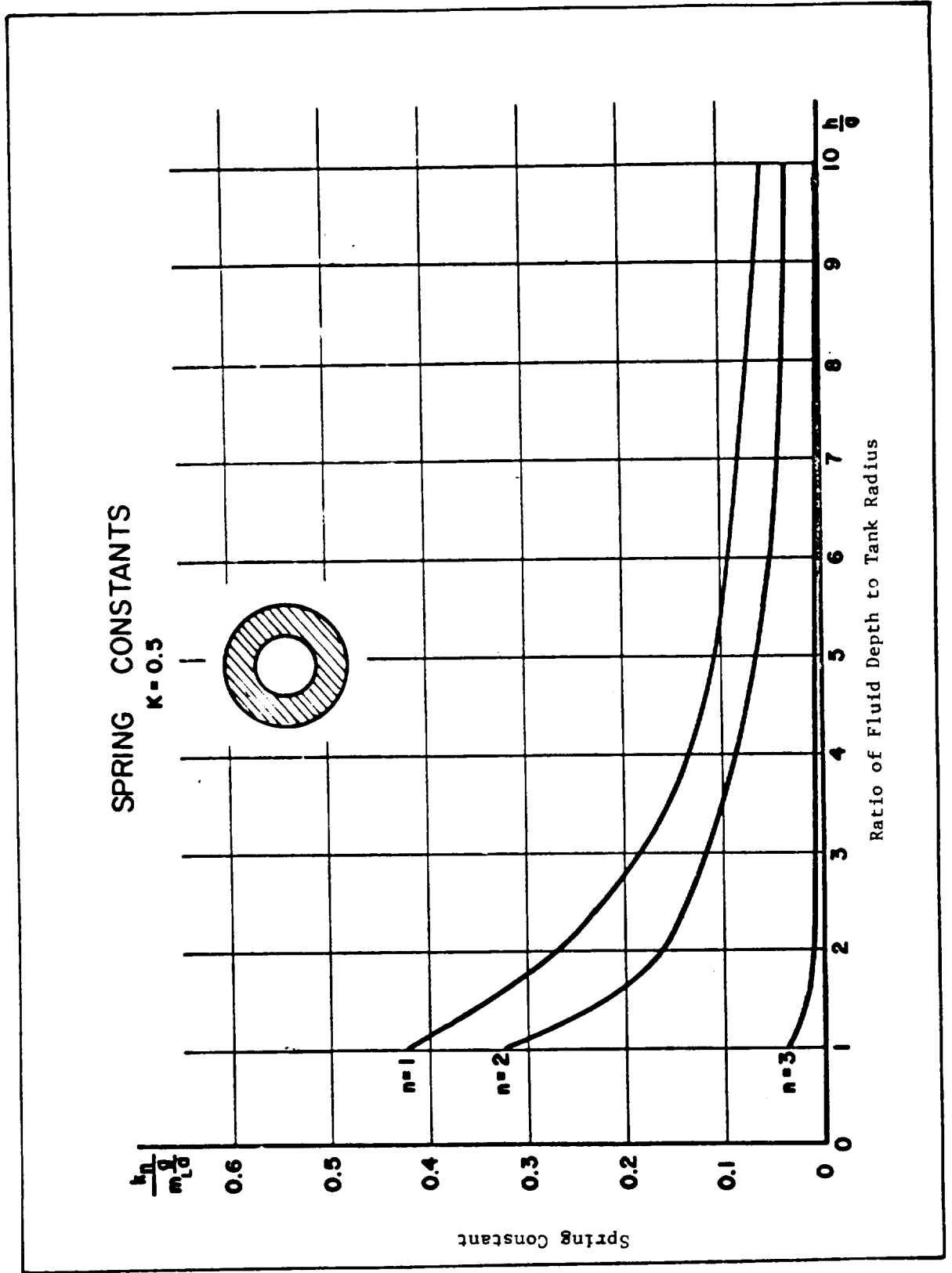


Figure 4-31. Model Element Graph





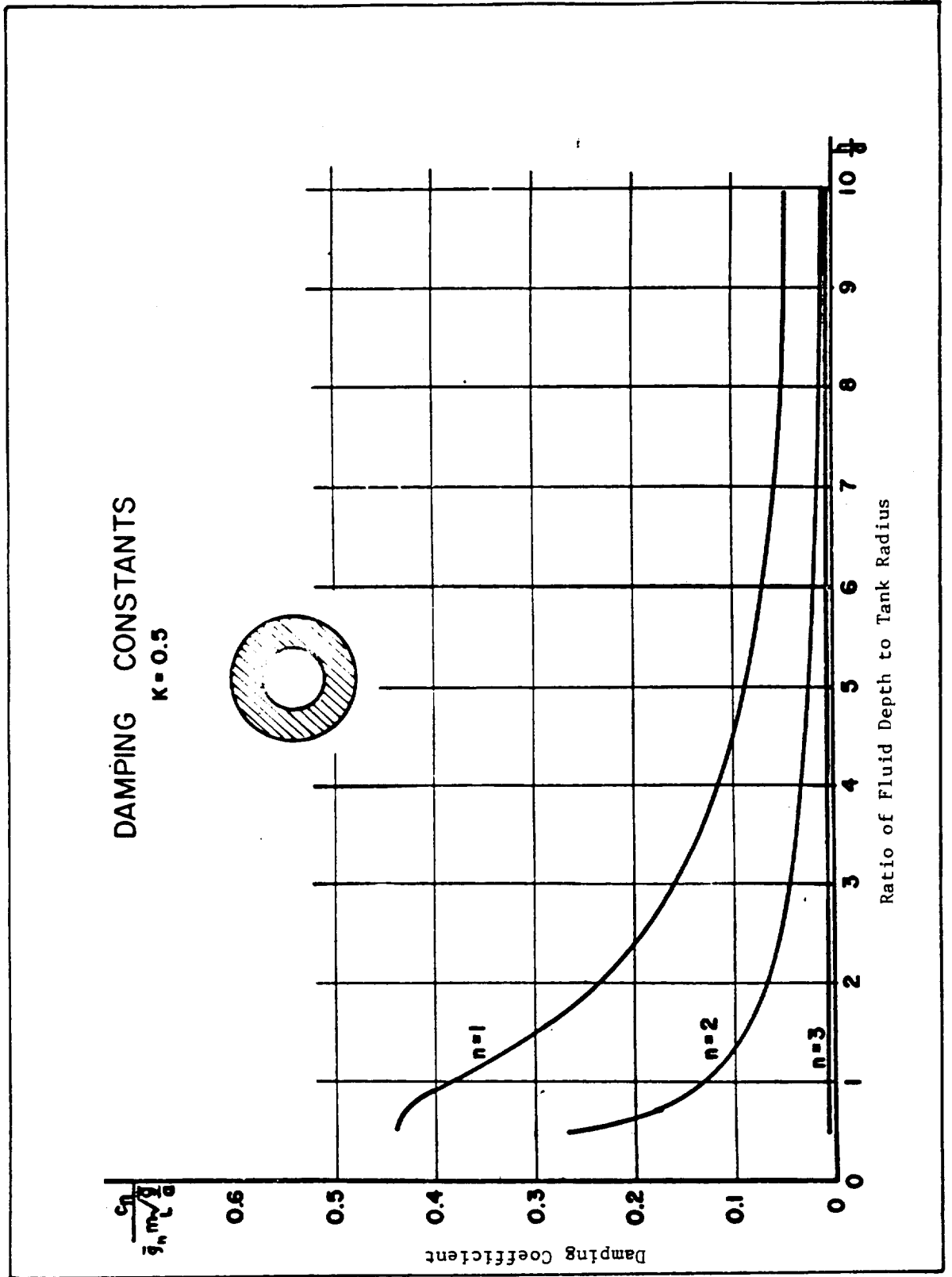


Figure 4-32. Model Element Graph

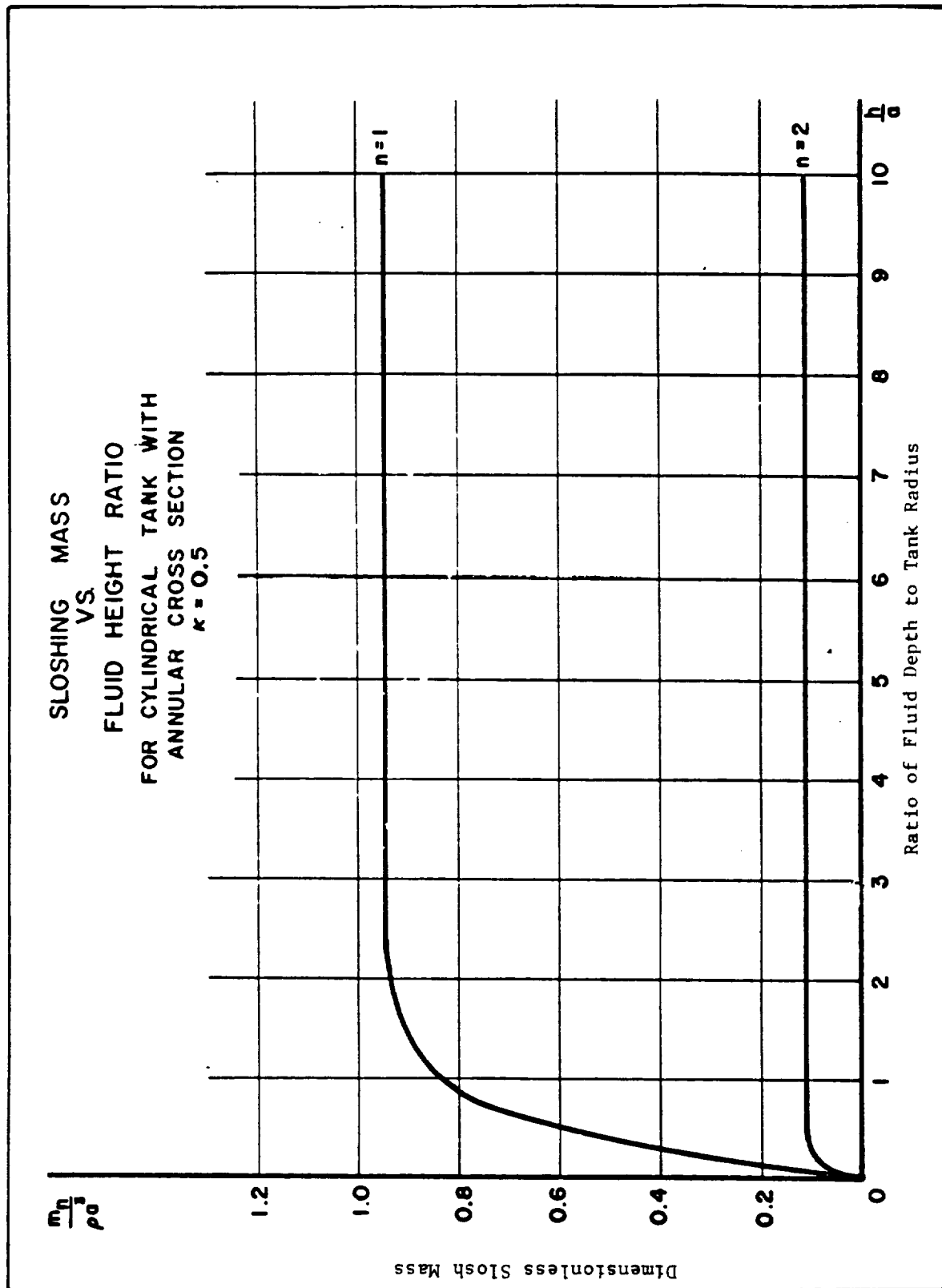


Figure 4-33. Model Element Graph

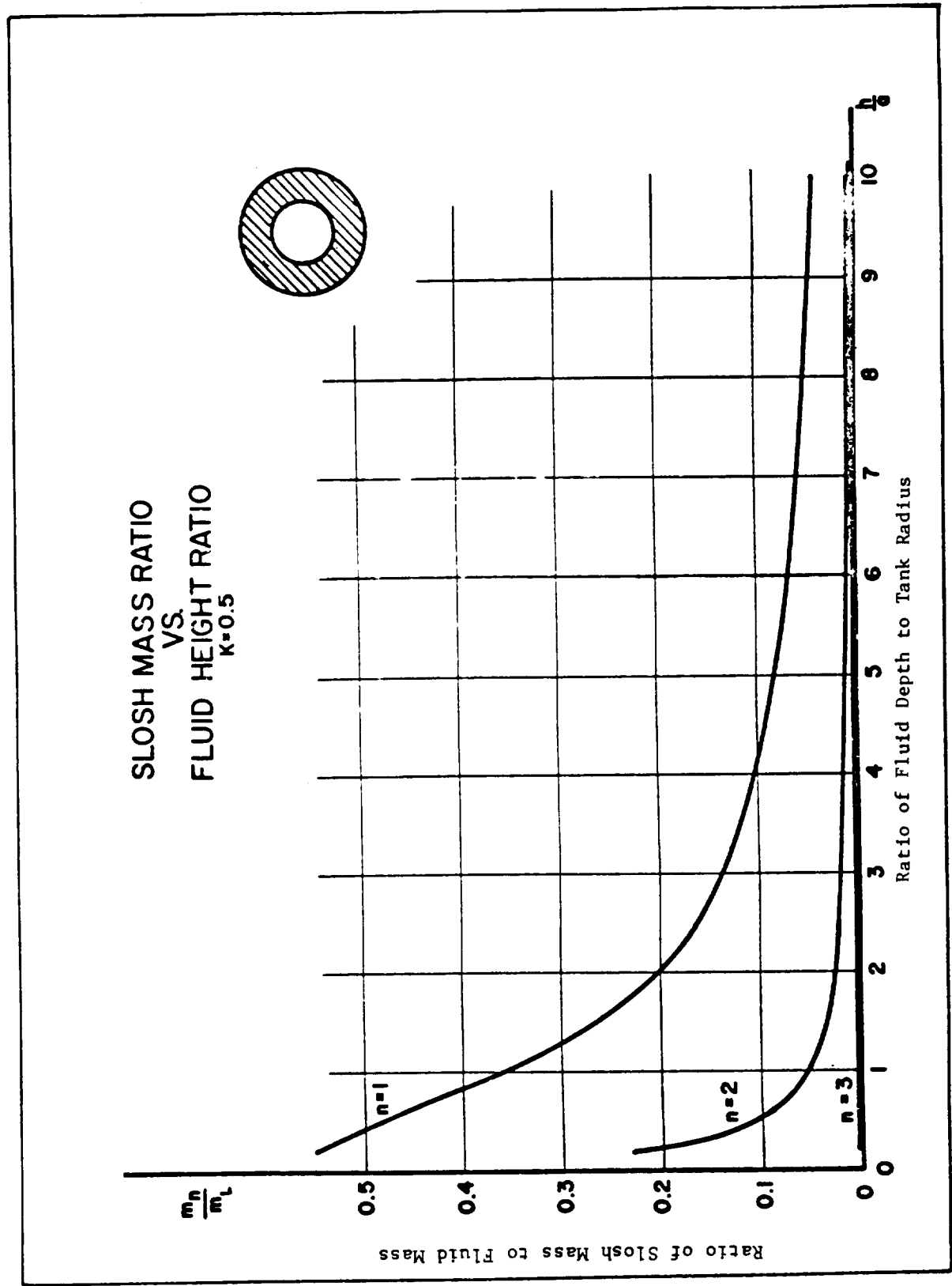


Figure 4-34. Model Element Graph

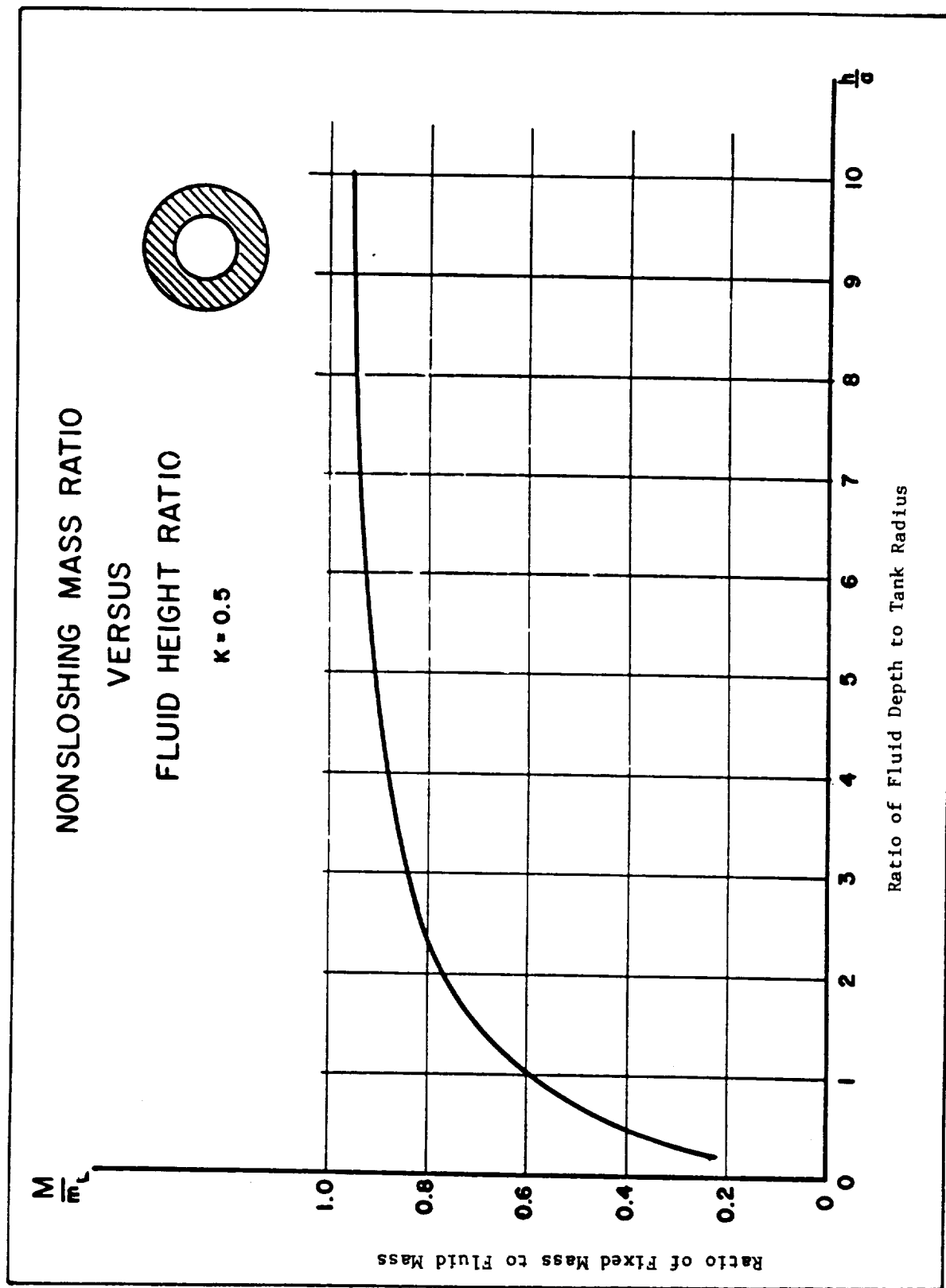


Figure 4-35. Model Element Graph

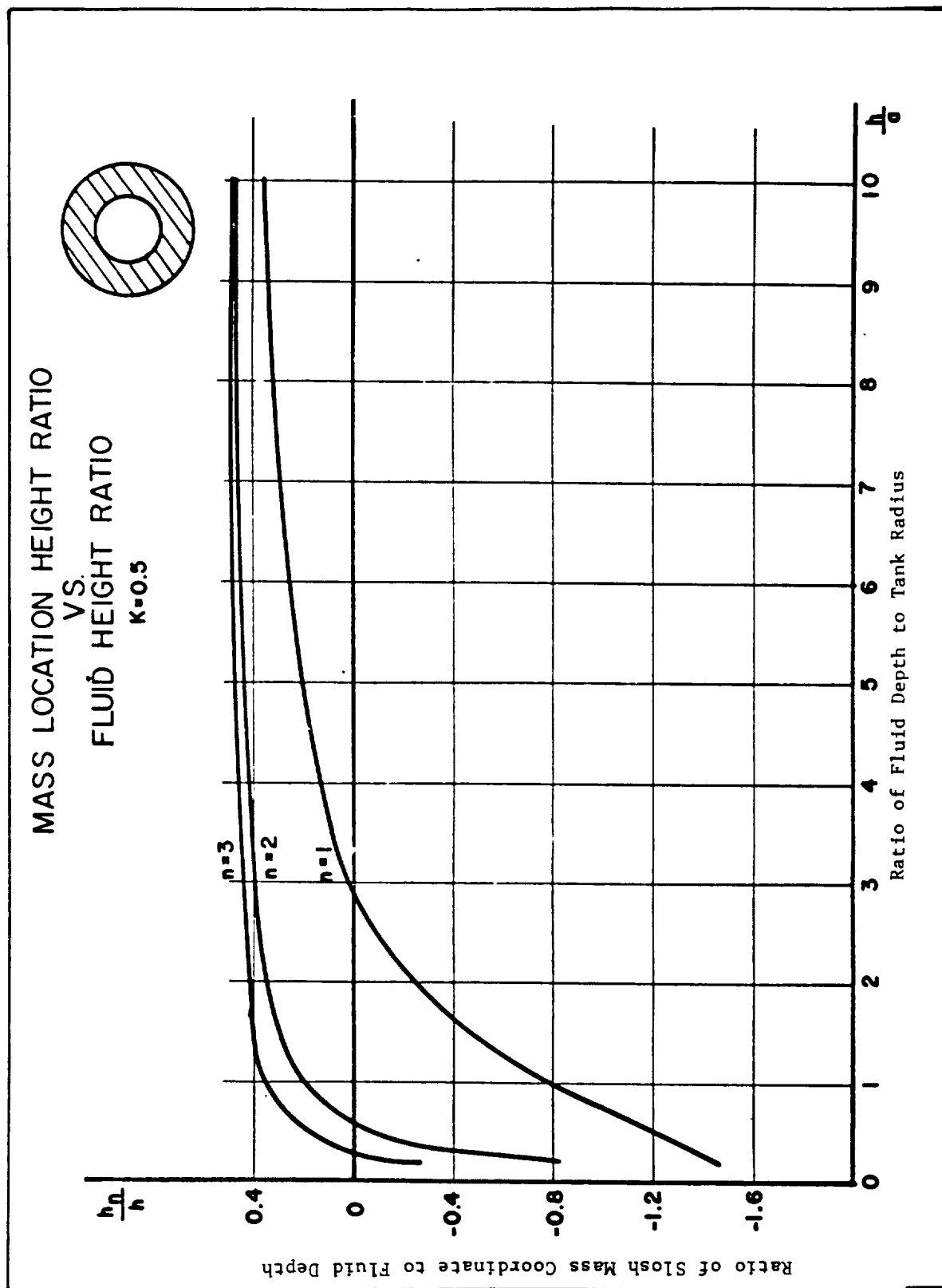


Figure 4-36. Model Element Graph

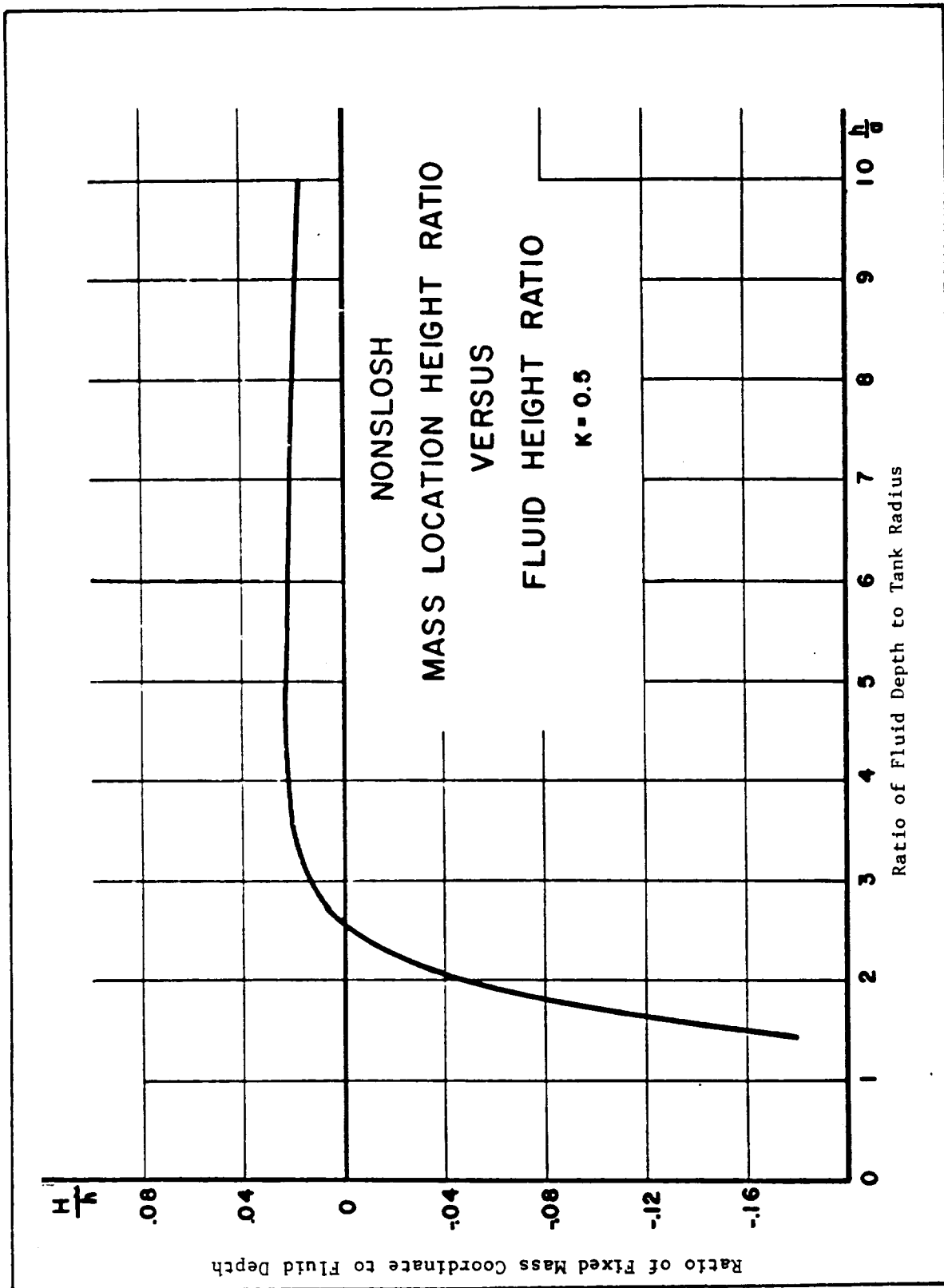


Figure 4-37. Model Element Graph

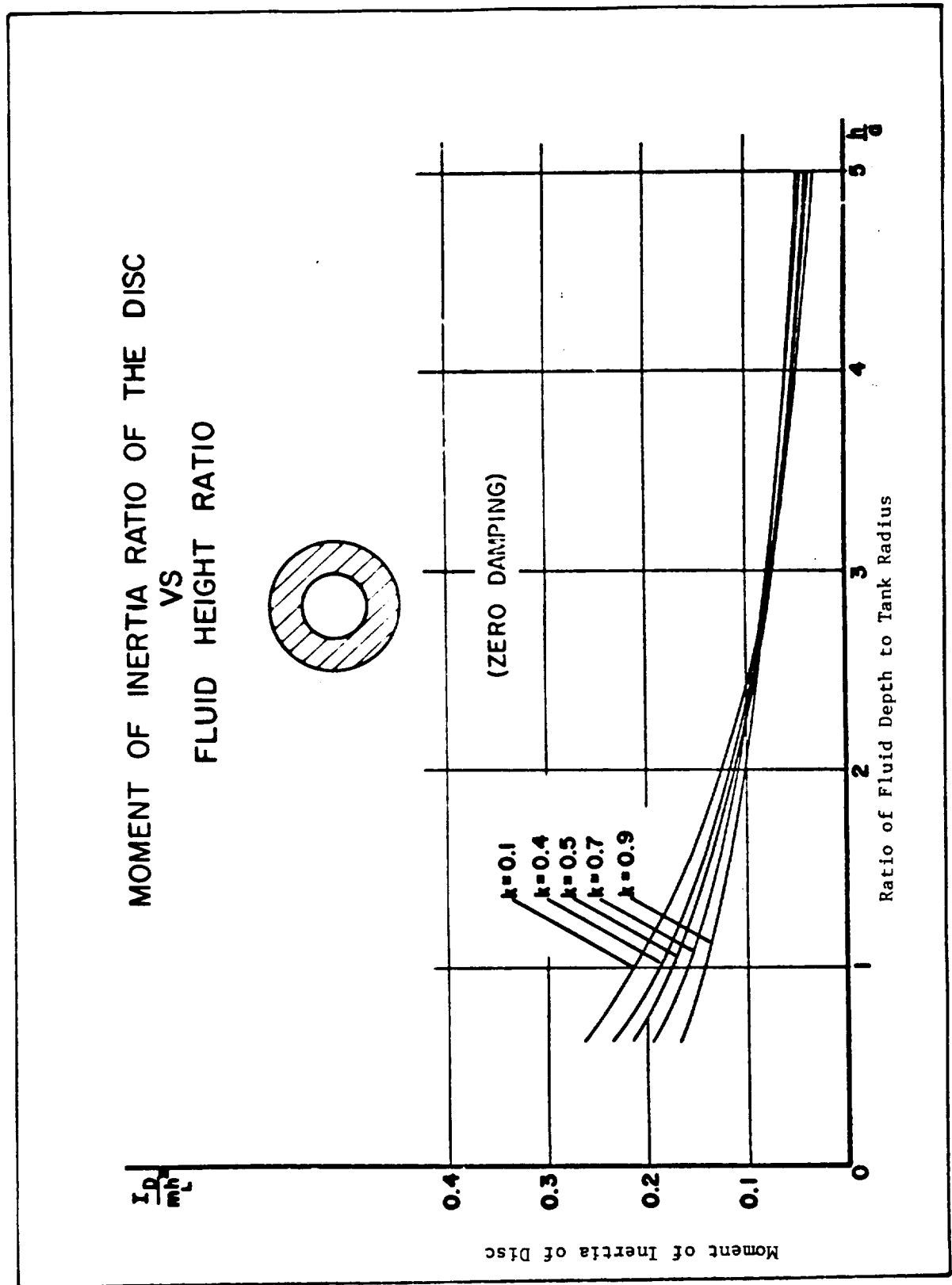


Figure 4-38. Model Element Graph

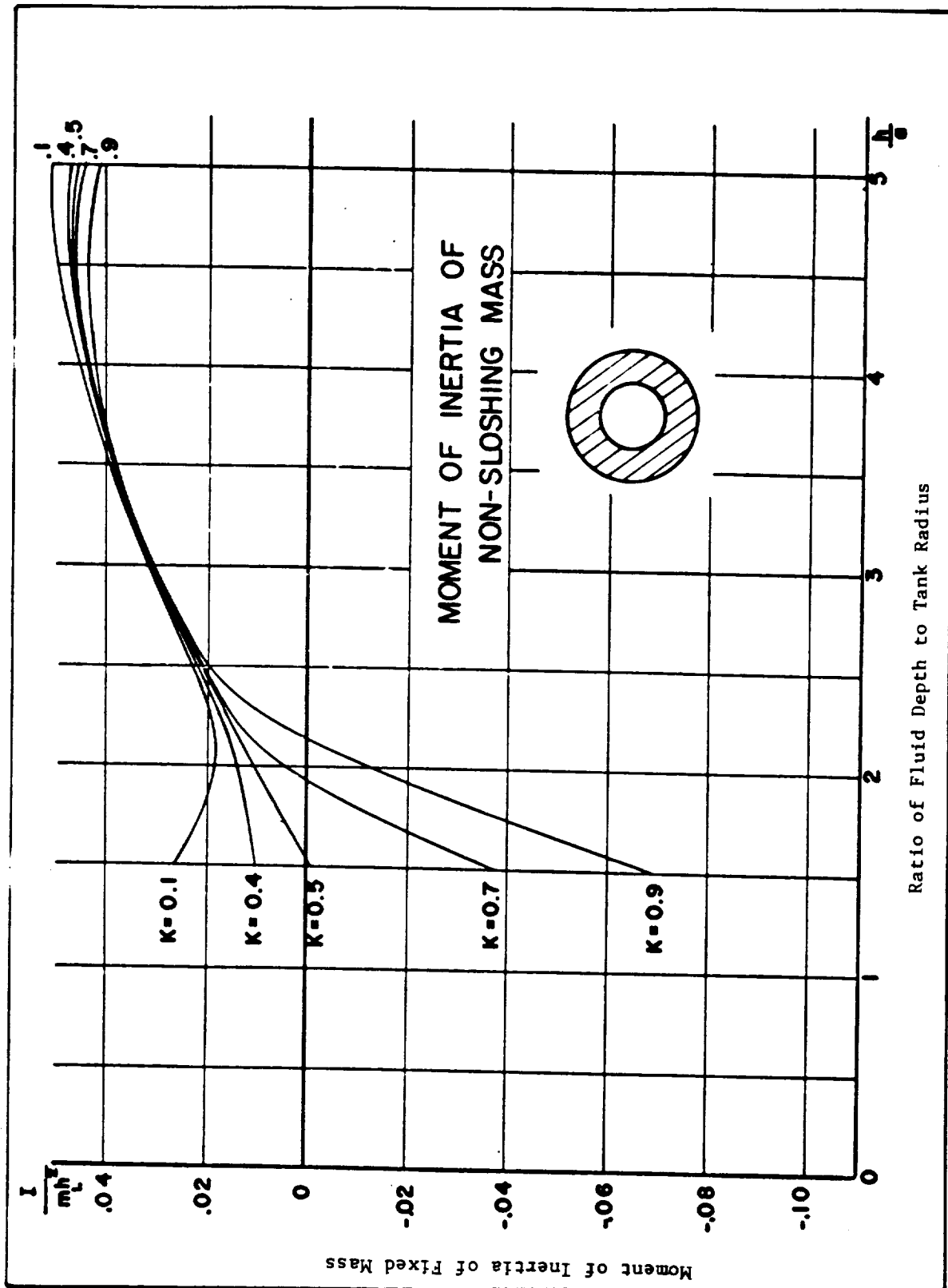


Figure 4-39. Model Element Graph



Table 4-41. Model Analysis

| Annular Tank   | Pendulum Model |
|--|----------------|
| Excitation: Harmonic Translation and/or Pitching   |                |
| <p>Figure 4-40 shows a diagram of the pendulum model used in representing the dynamic response of a liquid in an annular tank when subjected to harmonic translation in the x-direction and/or pitching about the y-axis.</p>  |                |
| <u>Coordinate System:</u>  |                |
| The origin is located at the center of gravity of the undisturbed liquid.  |                |
| <u>Model Description:</u>  |                |
| The components of the system are as follows:   |                |
| <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the <math>z</math>-axis at a distance <math>H</math> below the coordinate origin.</li> <li>2. A set of movable masses <math>m_n</math> is distributed along the <math>z</math>-axis when the tank is at rest. These modal masses are pendulums having massless lever arms of length <math>L_n</math> attached to the <math>z</math>-axis at distances <math>H_n</math> above the origin. They are constrained by dashpots having viscous damping coefficients <math>c_n</math> to remain approximately in the <math>xy</math>-plane and to move approximately parallel to the <math>x</math>-axis. Angular displacements of the pendulum with respect to the tank (<math>z</math>-axis) are denoted by <math>\lambda_n</math>.</li> <li>3. A massless disc having a moment of inertia <math>I_d</math> is located at the coordinate origin. Its motion is confined to rotation about the <math>y</math>-axis and is cushioned by a dashpot having a viscous damping coefficient <math>c_d</math>. The angular displacement of the disc relative to the tank is defined by <math>\psi</math>.</li> </ol> |                |
| <u>Equations of Motion:</u>  |                |
| The equations, obtained through Lagrange's equations, are as follows:  |                |
| 1. Force Equation:   |                |
| $F_x = -M(\ddot{x} + H\ddot{\theta}) - \sum_{n=1}^{\infty} m_n \left[ \ddot{x} + L_n \ddot{\lambda}_n + (H_n - L_n)\ddot{\theta} \right]$  |                |
| 2. Moment Equation:  |                |
| $M_y = - (I + MH^2)\ddot{\theta} - I_d(\ddot{\theta} + \ddot{\psi}) + g \sum_{n=1}^{\infty} m_n L_n \lambda_n - \sum_{n=1}^{\infty} m_n (H_n - L_n) \cdot \left[ L_n \ddot{\lambda}_n + (H_n - L_n)\ddot{\theta} \right]$  |                |

Table 4-41. Model Analysis (continued)

| Annular Tank   | Pendulum Model |
|--|----------------|
| Excitation: Harmonic Translation and/or Pitching   |                |
| <p data-bbox="201 426 678 457"><u>Equations of Motion (continued):</u></p> <p data-bbox="277 470 545 501">3. Disc Equation:</p> $I_d (\ddot{\theta} + \ddot{\psi}) + c_d \dot{\psi} = 0$ <p data-bbox="277 602 634 634">4. Slosh-Mass Equation:</p> $m_n \left[ \ddot{x} + L_n \ddot{\lambda}_n + (H_n - L_n) \ddot{\theta} \right] + m_n \bar{g}_n \omega_n L_n \dot{\lambda}_n - m_n g \theta = 0$ <p data-bbox="201 751 1341 806">From these equations, the model force in the x-direction and the moment about the y-axis can be found (see Table 4-42).</p> |                |

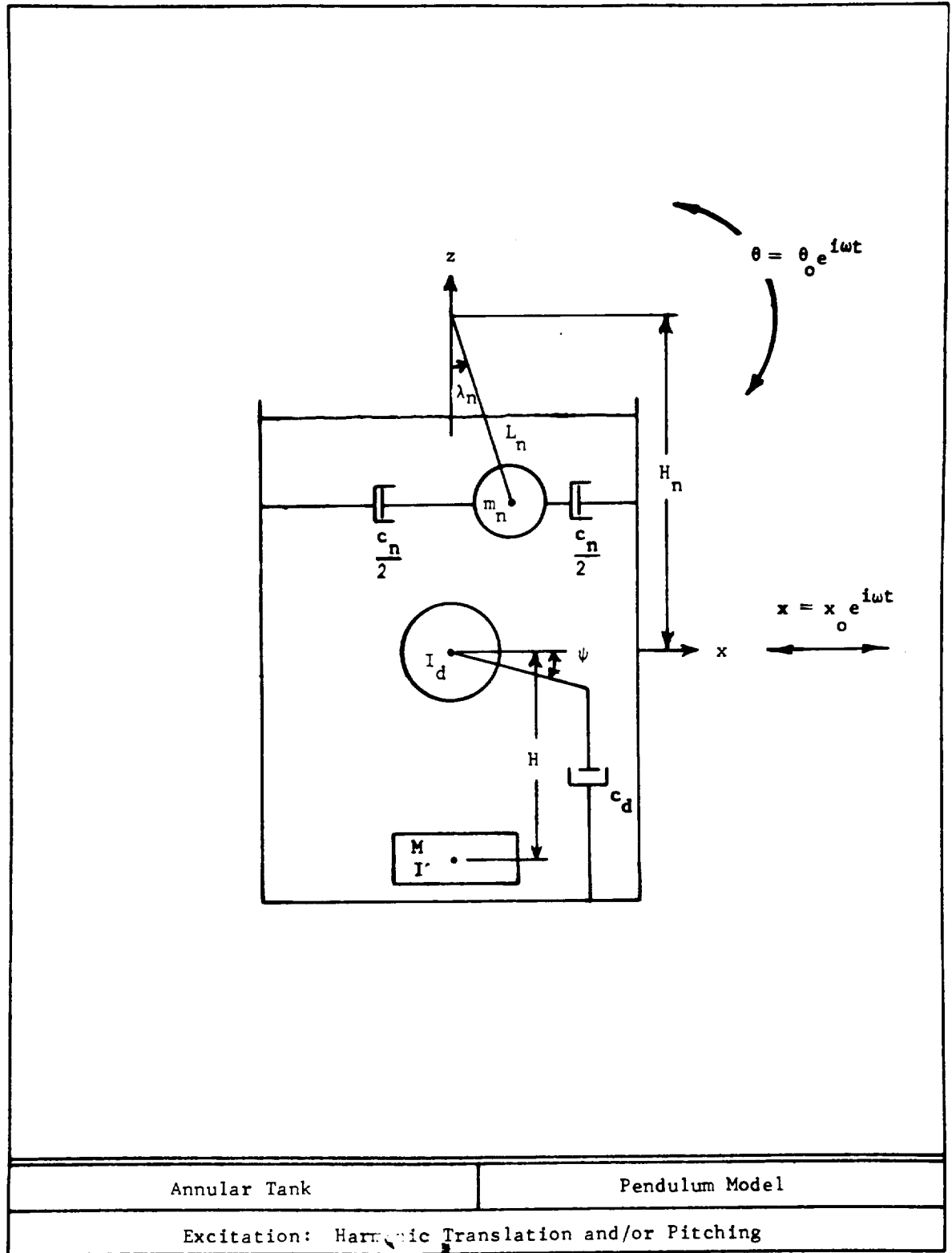


Figure 4-40. Equivalent Mechanical Model



Table 4-42. Model Force and Moment Resultants

| Annular Tank  | Pendulum Model |
|---|----------------|
| Excitation: Harmonic Translation and/or Pitching  |                |
| Translation in the x-direction, $x = x_o e^{i\omega t}$ .   |                |
| 1. $F_x = m_L \omega^2 x_o e^{i\omega t} \left[ 1 + \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) \right]$   |                |
| 2. $M_y = m_L h \omega^2 x_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{H_n - L_n}{h} + \frac{g}{h\omega^2} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$   |                |
| Pitching about the y-axis, $\theta = \theta_o e^{i\omega t}$ .  |                |
| 1. $F_x = -m_L h \omega^2 \theta_o e^{i\omega t} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{H_n - L_n}{h} + \frac{g}{h\omega^2} \right) \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right)$   |                |
| 2. $M_y = -m_L h^2 \omega^2 \theta_o e^{i\omega t} \left[ \frac{I}{m_L h^2} + \frac{M^2}{m_L h^2} + \frac{I_d}{m_L h^2} \left( 1 - \frac{\omega^2 I_d^2}{c_d^2 + \omega^2 I_d^2} \right) + \sum_{n=0}^{\infty} \frac{m_n (H_n - L_n)^2}{m_L h^2} \right. \\ \left. + \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{H_n - L_n}{h} + \frac{g}{h\omega^2} \right)^2 \left( \frac{1}{\eta_n^2 - 1 + i\bar{g}_n \eta_n} \right) \right] + i\omega \theta_o e^{i\omega t} \left( \frac{c_d \omega I_d^2}{c_d^2 + \omega^2 I_d^2} \right)$ |                |



Table 4-43. Model Elements

| Annular Tank                                     |   | Pendulum Model |  |
|--|---|----------------|--|
| Excitation: Harmonic Translation and/or Pitching |   |                |  |
| Natural Frequency                                | $\omega_n^2 = \frac{g}{a} \xi_n \tanh \kappa_n$   |                |  |
| Pendulum Length                                  | $L_n = \frac{g}{\omega_n^2}$ (fig. 4-41)  |                |  |
| Damping Coefficient of Slosh Mass                | $c_n = m_n \bar{g}_n \omega_n$  |                |  |
| Damping Coefficient                              | $c_d = \bar{c} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   |                |  |
| Ratio of Slosh Mass to Fluid Mass                | $\frac{m_n}{m_L} = \frac{A_n [2/\pi \xi_n - k C_1(\sigma_n)] \tanh \kappa_n}{(1-k^2) \kappa_n}$   |                |  |
| Ratio of Fixed Mass to Fluid Mass                | $\frac{M}{m_L} = 1 - \sum_{n=0}^{\infty} \frac{m_n}{m_L}$   |                |  |
| Ratio of Pendulum Mass Coordinate to Fluid Depth | $\frac{ h_n - L_n }{h} = \frac{1}{2} \left[ 1 - \frac{4}{\kappa_n} \tanh\left(\frac{\kappa_n}{2}\right) \right]$ (fig. 4-42)  |                |  |
| Ratio of Fixed Mass Coordinate to Fluid Depth    | $\left  \frac{h}{h} \right  = \frac{m_L}{M} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h_n - L_n}{h} \right)$  |                |  |
| Moment of Inertia of Solidified Fluid            | $\frac{I_s}{m_L h^2} = \left( \frac{a}{h} \right)^2 \left[ \frac{1}{12} \left( \frac{h}{a} \right)^2 + \frac{1}{4} \right]$   |                |  |
| Moment of Inertia of Disc ( $c_d = 0$ )          | $\frac{I_d}{m_L h^2} = 4 \left( \frac{a}{h} \right)^2 \sum_{n=0}^{\infty} \frac{A_n [2/\pi \xi_n - k C_1(\sigma_n)] [1 - (2/\kappa_n) \tanh(\kappa_n/2)]}{(1-k^2) \xi_n}$             |                |  |
| Moment of Inertia of Disc ( $c_d \neq 0$ )       | $\frac{I_d}{m_L h^2} = \frac{I_s - \bar{I}}{m_L h^2} \left[ 1 + \frac{\bar{c}^2}{\omega^2 (I_s - \bar{I})^2} \right]$   |                |  |
| Moment of Inertia of Fixed Mass                  | $\frac{I}{m_L h^2} = \frac{I_s}{m_L h^2} - \frac{I_d}{m_L h^2} - \frac{M}{m_L} \left( \frac{h}{h} \right) - \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left[ \frac{h_n - L_n}{h} \right]^2$ |                |  |





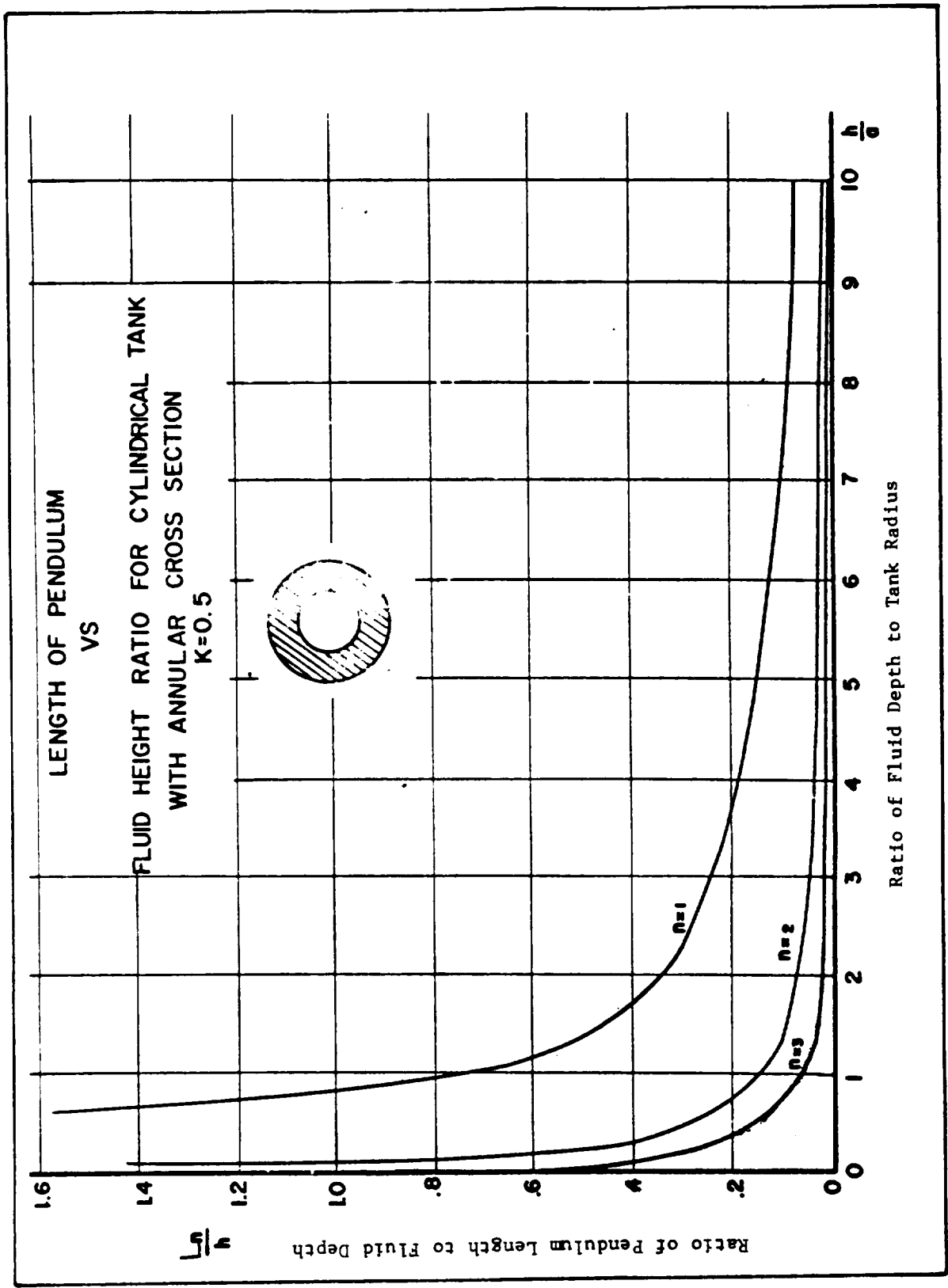


Figure 4-41. Model Element Graph



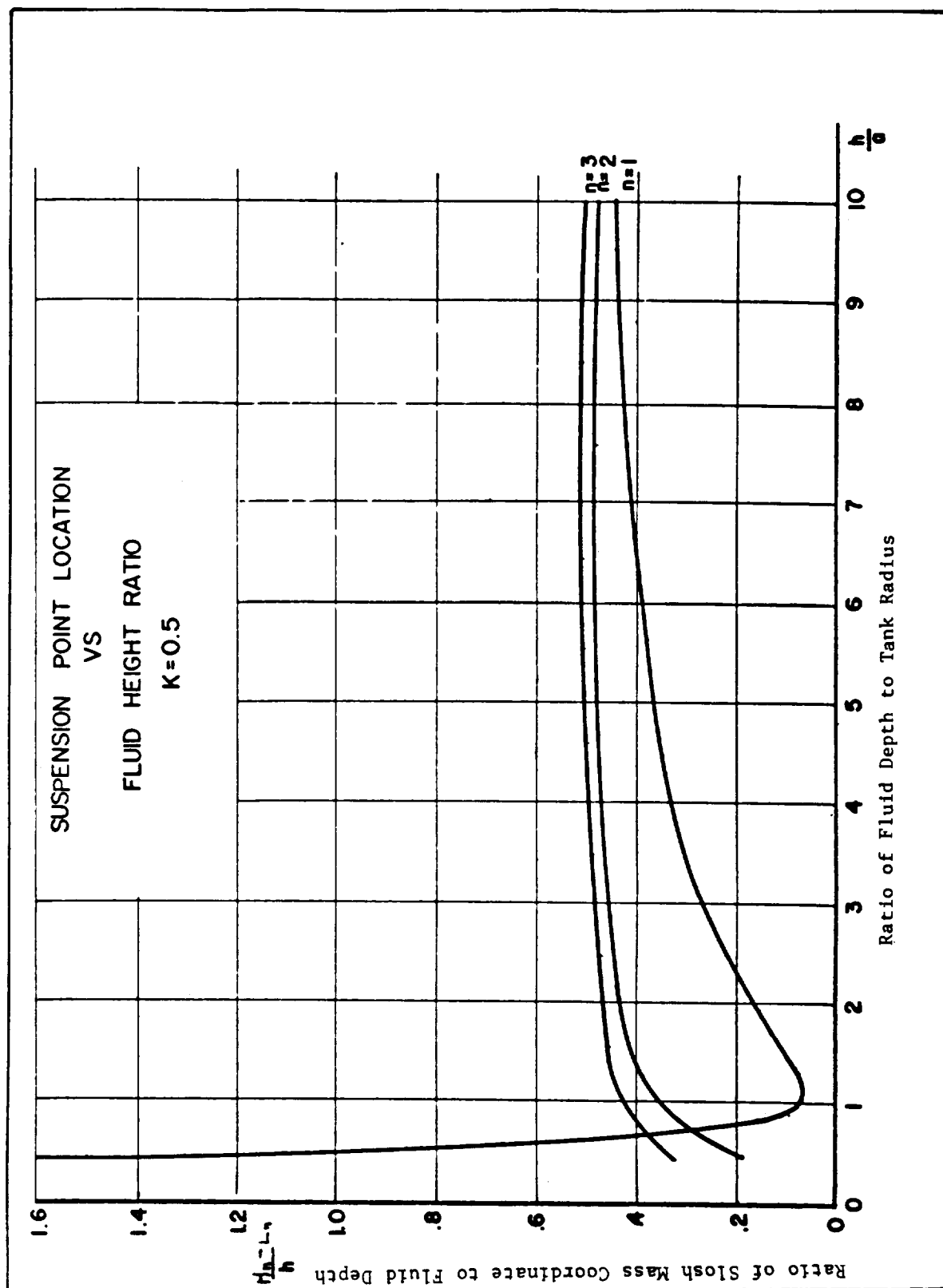


Figure 4-42. Model Element Graph



## IV. Rigid Tanks

## 4.7 Annular-Sector Tank

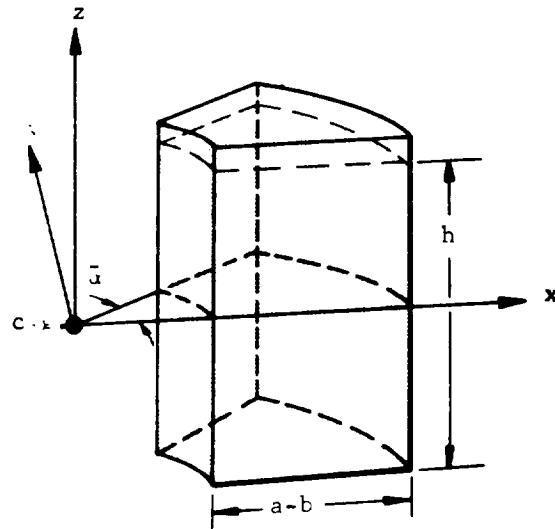
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## IV. Rigid Tanks

## 4.7 Annular-Sector Tank



Container: The tank is a sector of an annular tank having an inner radius  $r=b$  and an outer radius  $r=a$  and is filled with a liquid to a depth  $h$ .

Coordinate System: The origin is located at the center of gravity of the undisturbed fluid which would be contained in an annular tank generated by revolving the annular-sector tank about the  $z$ -axis. The  $x$ -axis must lie in the sector wall.

Reference: (4)

Comments: The results given in this section are not valid for  $\bar{\alpha} = \frac{\pi}{2}, \frac{3\pi}{2}$ .





Table 4-44. Boundary Conditions, Velocity Potential, and Natural Frequency

| Annular-Sector Tank   | Excitation: Harmonic Translation  |
|---|---|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .   |   |
| 1. Boundary conditions:   |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a,b} = i\omega x_0 e^{i\omega t} \cos \phi$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$                  |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$  | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = 0$ |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\alpha} = -i\omega x_0 e^{i\omega t} \sin \alpha$   |   |
| 2. Velocity potential:  |   |
| $\phi = i\omega x_0 e^{i\omega t} \left[ r \cos \phi + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a b C_{mn}(\rho_{mn}) \cos \phi_m \cosh(\kappa_{mn} + \zeta_{mn})}{(\eta_{mn}^2 - 1) \cosh \kappa_{mn}} \right]$ |   |
| 3. Natural angular frequency:   |   |
| $\omega_{mn}^2 = \frac{g}{a} \xi_{mn} \tanh \kappa_{mn}$  |   |
| where $\xi_{mn}$ are roots of $\Delta_{m/2\alpha}(\xi_{mn}) = J'_{m/2\alpha}(\xi_{mn}) Y'_{m/2\alpha}(k\xi_{mn}) - J'_{m/2\alpha}(k\xi_{mn}) Y'_{m/2\alpha}(\xi_{mn}) = 0$ .  |   |



Table 4-44. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

| Annular-sector Tank  | Excitation: Harmonic Translation  |
|--|---|
| Translation in the y-direction, $y = y_0 e^{i\omega t}$ .  |   |
| 1. Boundary conditions:  |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a,b} = i\omega y_0 e^{i\omega t} \sin \phi$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$  |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$   | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = i\omega y_0 e^{i\omega t}$ |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\alpha} = i\omega y_0 e^{i\omega t} \cos \alpha$   |   |
| 2. Velocity potential:   |   |
| $\phi = i\omega y_0 e^{i\omega t} \left[ r \sin \phi + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c_{mn}^b C_{mn}(\rho_{mn}) \cos \bar{\phi}_m \cosh(\kappa_{mn} + \zeta_{mn})}{(\eta_{mn}^2 - 1) \cosh \kappa_{mn}} \right]$ |   |
| 3. Natural angular frequency:  |   |
| $\omega_{mn}^2 = \frac{g}{a} \xi_{mn} \tanh \kappa_{mn}$   |   |
| where $\xi_{mn}$ are roots of $\Delta_{m/2\alpha}(\xi_{mn}) = J'_{m/2\alpha}(\xi_{mn}) Y'_{m/2\alpha}(\kappa \xi_{mn}) - J'_{m/2\alpha}(\kappa \xi_{mn}) Y'_{m/2\alpha}(\xi_{mn}) = 0$ .   |   |

Table 4-45. Liquid Force and Moment Resultants

| Annular-Sector Tank  | Excitation: Harmonic Translation |
|--|----------------------------------|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$ .  |                                  |
| <p>1. <math display="block">\begin{pmatrix} F_x \\ F_y \end{pmatrix} = + m_L \omega^2 x_0 e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \begin{pmatrix} 1+ \\ 0 \end{pmatrix} \frac{2 a b_{mn} \tanh \kappa_{mn}}{\bar{\alpha} a (1-k^2)(\eta_{mn}^2 - 1) \kappa_{mn}} \begin{pmatrix} (-1)^{m+1} \sin \bar{\alpha} \\ 1 - (-1)^m \cos \bar{\alpha} \end{pmatrix} \begin{bmatrix} N(\xi_{mn}) \\ N_o(\xi_{mn}) \end{bmatrix} + \right. \\ \left. \frac{\bar{\alpha}^2}{\pi_m^2 - \bar{\alpha}^2} \left( \frac{2}{\pi \xi_{mn}} - k C_m / 2 \alpha_{mn} \right) \right\}</math></p>   |                                  |
| <p>2. <math display="block">\begin{pmatrix} M_x \\ M_y \end{pmatrix} = + m_L a \omega^2 x_0 e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \left\{ \frac{(1+k^2)}{4(h/a)} \right\} \left( \frac{\sin \bar{\alpha} / \bar{\alpha}}{1 + \sin \bar{\alpha} \cos \bar{\alpha} / \bar{\alpha}} \right) + \frac{2 \pi a b_{mn}}{\bar{\alpha}^2 (1-k^2)(\eta_{mn}^2 - 1) \xi_{mn}} \right. \\ \left. \begin{pmatrix} 1 - (-1)^m \cos \bar{\alpha} \\ (-1)^{m+1} \sin \bar{\alpha} \end{pmatrix} \left[ \tanh \kappa_{mn} + \frac{2}{\kappa_{mn}} \left( \frac{1}{\cosh \kappa_{mn}} - 1 \right) \right] \begin{bmatrix} N_o(\xi_{mn}) + \frac{\bar{\alpha}^2}{\pi_m^2 - \bar{\alpha}^2} \left( \frac{2}{\pi \xi_{mn}} - k C_m / 2 \alpha_{mn} \right) \end{bmatrix} \right. \\ \left. + \frac{2 \bar{\alpha}^2 \xi_{mn}^2 N(\xi_{mn})}{(\pi_m^2 - \bar{\alpha}^2) \kappa_{mn} \cosh \kappa_{mn}} \right\} + \frac{2 m_L g a (1-k^3)}{3 \bar{\alpha} (1-k^2)} \begin{pmatrix} 1 - \cos \bar{\alpha} \\ \sin \bar{\alpha} \end{pmatrix}</math></p> |                                  |

Table 4-45. Liquid Force and Moment Resultants (continued)

| Annular Sector Tank                                     |  | Excitation: Harmonic Translation   |
|---|--|--|
| Translation in the y-direction, $y = y_0 e^{i\omega t}$ |  |  |
| 1.  | $\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m_L \omega^2 y_0 e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{2c b_{mn} \tanh \kappa_{mn}}{\bar{\alpha} \alpha (1-k^2)(\eta_{mn}^2 - 1) \kappa_{mn}} \begin{pmatrix} (-1)^{m+1} \sin \bar{\alpha} \\ 1 - (-1)^m \cos \bar{\alpha} \end{pmatrix} \left[ \frac{-2}{\pi m^2 - \bar{\alpha}^2} \right] \begin{pmatrix} 2 \\ \pi \xi_{mn} \end{pmatrix} - \right.$ $\left. k C_{m/2\alpha}(\sigma_{mn}) + N_o(\xi_{mn}) \right\}$   | $\begin{pmatrix} 1 - (-1)^m \cos \bar{\alpha} \\ (-1)^{m+1} \sin \bar{\alpha} \end{pmatrix}$ |
| 2.  | $\begin{pmatrix} M_x \\ M_y \end{pmatrix} = m_L a \omega^2 y_0 e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \left\{ \frac{(1+k^2)}{4(h/a)} - \left( \frac{1 + \sin \bar{\alpha} \cos \bar{\alpha}/2}{\sin^2 \bar{\alpha}/\alpha} \right) \right\} \frac{c b_{mn}}{\bar{\alpha} \alpha (1-k^2)(\eta_{mn}^2 - 1) \xi_{mn}} \right.$ $\left. \left\{ \left[ \tanh \kappa_{mn} + \frac{2}{\kappa_{mn}} \left( \frac{1}{\cosh \kappa_{mn}} - 1 \right) \right] \left[ \frac{-2}{\pi m^2 - \bar{\alpha}^2} \right] \begin{pmatrix} 2}{\pi \xi_{mn}} - k C_{m/2\alpha}(\sigma_{mn}) + N_o(\xi_{mn}) \right] \right\} \right.$ $\left. + \frac{2 \bar{\alpha}^2 \xi_{mn}^2 N_2(\xi_{mn})}{(\pi m^2 - \bar{\alpha}^2) \kappa_{mn} \cosh \kappa_{mn}} \right\} + \frac{2 m_L g a (1-k^3)}{3 \bar{\alpha} (1-k^2)} \begin{pmatrix} 1 - \cos \bar{\alpha} \\ \sin \bar{\alpha} \end{pmatrix}$ |  |

Table 4-46. Boundary Conditions, Velocity Potential, and Natural Frequency

| Annular Sector Tank   | Excitation: Harmonic Pitching   |
|---|---|
| Pitching about the x-axis, $\chi = \chi_0 e^{i\omega t}$ .  |   |
| 1. Boundary conditions:   |   |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a,b} = -i\omega z \chi_0 e^{i\omega t} \sin \phi$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = i\omega z \chi_0 e^{i\omega t} \sin \phi$       |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=th/2} = 0$   | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = -i\omega z \chi_0 e^{i\omega t}$ |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\bar{\alpha}} = -i\omega \chi_0 e^{i\omega t} \cos \bar{\alpha}$  |   |
| 2. Velocity potential:  |   |
| $\phi = -i\omega \chi_0 e^{i\omega t} \left[ rz \sin \phi - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m/2\alpha}(\rho_{mn}) \cos \bar{\phi}_m (C_{mn} \cosh \zeta_{mn} + D_{mn} \sinh \zeta_{mn}) \right]$ |   |
| 3. Natural angular frequency:   |   |
| $\omega_{mn}^2 = \frac{g}{a} \xi_{mn} \tanh \kappa_{mn}$  |   |
| where $\xi_{mn}$ are roots of $\Delta_{m/2\alpha}(\xi_{mn}) = J'_{m/2\alpha}(\xi_{mn}) Y'_{m/2\alpha}(\kappa \xi_{mn}) - J'_{m/2\alpha}(\kappa \xi_{mn}) Y'_{m/2\alpha}(\xi_{mn}) = 0$ .                    |   |

Table 4-46. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

| Annular-Sector Tank   | Excitation: Harmonic Pitching  |
|---|--|
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ .  |  |
| 1. Boundary conditions:   |  |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a,b} = -i\omega z \theta_0 e^{i\omega t} \cos \phi$   | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=h/2} = i\omega r \theta_0 e^{i\omega t} \cos \phi$ |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=h/2} = 0$  | (d) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0} = 0$                          |
| (e) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=\bar{\alpha}} = i\omega z \theta_0 e^{i\omega t} \sin \bar{\alpha}$   |  |
| 2. Velocity potential:  |  |
| $\phi = -i\omega \theta_0 e^{i\omega t} \left[ rz \cos \phi - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m/2\alpha}(\rho_{mn}) \cos \bar{\phi}_m (A_{mn} \cosh \zeta_{mn} + B_{mn} \sinh \zeta_{mn}) \right]$ |  |
| 3. Natural angular frequency:   |  |
| $\omega_{mn}^2 = \frac{g}{a} \zeta_{mn} \tanh \kappa_{mn}$  |  |
| where $\zeta_{mn}$ are roots of $\Lambda_{m/2\alpha}(\zeta_{mn}) = J'_{m/2\alpha}(\zeta_{mn}) Y'_{m/2\alpha}(\kappa \zeta_{mn}) - J'_{m/2\alpha}(\kappa \zeta_{mn}) Y'_{m/2\alpha}(\zeta_{mn}) = 0$ .         |  |

Table 4-47. Liquid Force and Moment Resultants

| Annular-Sector Tank   | Excitation: Harmonic Pitching |
|---|-------------------------------|
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ and pitching about the x-axis, $\chi = \chi_0 e^{i\omega t}$ .   |                               |
| 1. $F_y = -m_L g \begin{pmatrix} 0 \\ \chi_0 \end{pmatrix} e^{i\omega t} - 4m_L \omega^2 \begin{pmatrix} \theta_0 \\ \chi_0 \end{pmatrix} e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{A_{mn}}{C_{mn}} \right) \frac{[1 - (-1)^m \cos \bar{\alpha}] \sinh(\kappa_{mn}/2)}{\bar{\alpha} a (1 - k^2) \kappa_{mn}} \left[ N_0(\xi_{mn}) + \right.$   |                               |
| $\left. \frac{\bar{\alpha}^2}{(\pi^2 m^2 - \bar{\alpha}^2)} \left( \frac{2}{\pi \xi_{mn}} - k C_{m/2\alpha}(\sigma_{mn}) \right) \right]$   |                               |
| 2. $M_x = m_L g a \begin{pmatrix} \theta_0 \\ \chi_0 \end{pmatrix} e^{i\omega t} \left[ \frac{1 + k^2}{4(h/a)} \right] \left( \frac{\sin^2 \bar{\alpha}/\bar{\alpha}}{1 - \sin \bar{\alpha} \cos \bar{\alpha}/\bar{\alpha}} \right) - m_L a \omega^2 \begin{pmatrix} \theta_0 \\ \chi_0 \end{pmatrix} e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \left( \frac{B_{mn}}{D_{mn}} \right) \left[ \cosh \frac{\kappa_{mn}}{2} - \frac{2}{\kappa_{mn}} \sinh \frac{\kappa_{mn}}{2} \right] \left[ \frac{\bar{\alpha}^2}{\pi^2 m^2 - \bar{\alpha}^2} \cdot \right. \right.$ |                               |
| $\left. \left( \frac{2}{\pi \xi_{mn}} - k C_{m/2\alpha}(\sigma_{mn}) \right) + N_0(\xi_{mn}) \right] + \frac{\bar{\alpha}^2 \xi_{mn}^2 N(\xi_{mn})}{(\pi^2 m^2 - \bar{\alpha}^2) \kappa_{mn}} \left[ \left( \frac{A_{mn}}{C_{mn}} \right) \cosh \frac{\kappa_{mn}}{2} - \left( \frac{B_{mn}}{D_{mn}} \right) \sinh \frac{\kappa_{mn}}{2} \right] \right\}$  |                               |
| $- \frac{2m_L g a (1 - k^3)(1 - \cos \bar{\alpha})}{3\bar{\alpha}(1 - k^2)}$  |                               |



Table 4-47. Liquid Force and Moment Resultants (continued)

| Annular-Sector Tank  | Excitation: Harmonic Pitching |
|--|-------------------------------|
| $3. \quad M_y = -m_L g a \begin{pmatrix} 0 \\ x_o \end{pmatrix} e^{i\omega t} \left[ \frac{1+k^2}{4(h/a)} \right] \left( \frac{1 + \sin \bar{\alpha} \cos \bar{\alpha} / \bar{\alpha}}{\sin^2 \bar{\alpha} / \bar{\alpha}} \right) + m_L a_\omega^2 \begin{pmatrix} 0 \\ x_o \end{pmatrix} e^{i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cdot$ $\left[ \left( \frac{(1+k^2)}{(1+k^2) \sin^2 \bar{\alpha} / 8\bar{\alpha}} \left[ 1 + (\sin \bar{\alpha} \cos \bar{\alpha}) / \bar{\alpha} \right] / 8 - (h/a)^2 / 12 \right) + \frac{2(-1)^{m+1} \sin \bar{\alpha}}{\bar{\alpha} a^2 (1-k^2) \xi_{mn}} \left\{ \begin{pmatrix} B_{mn} \\ D_{mn} \end{pmatrix} \left[ \cosh \frac{\kappa_{mn}}{2} - \frac{2}{\kappa_{mn}} \sinh \frac{\kappa_{mn}}{2} \right] \right\} \right.$ $\left[ \frac{\bar{\alpha}^2}{\pi \xi_{mn}^2 - \bar{\alpha}^2} \left( \frac{2}{\pi \xi_{mn}} - k C_{m/2\alpha}(\sigma_{mn}) + N_o(\xi_{mn}) \right) + \frac{\bar{\alpha}^2 \xi_{mn}^2 N_2(\xi_{mn})}{(2\xi_{mn}^2 - \bar{\alpha}^2) \kappa_{mn}} \left[ \begin{pmatrix} A_{mn} \\ C_{mn} \end{pmatrix} \cosh \frac{\kappa_{mn}}{2} - \right.$ $\left. \left. \left. \begin{pmatrix} B_{mn} \\ D_{mn} \end{pmatrix} \sinh \frac{\kappa_{mn}}{2} \right] \right] \right] + \frac{2m_L g a (1-k^3) \sin \bar{\alpha}}{3\bar{\alpha} (1-k^2)}$ |                               |

Table 4-48. Boundary Conditions, Velocity Potential, and Natural Frequency

| Annular-Sector Tank  |  | Excitation: Harmonic Roll |
|--|--|---------------------------|
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .   |  |                           |
| 1. Boundary conditions:  |  |                           |
| (a) $\left( \frac{\partial \phi}{\partial r} \right)_{r=a,b} = 0$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h} = 0$   |                           |
| (c) $\left( \frac{1}{r} \frac{\partial \phi}{\partial \phi} \right)_{\phi=0,\bar{\alpha}} = i\omega r \phi_0 e^{i\omega t}$  | (d) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=0} = 0$ |                           |
| 2. Velocity potential:   |  |                           |
| $\phi = i\omega a^2 \phi_0 e^{i\omega t} \left\{ \left( \frac{r}{a} \right)^2 \left( \phi - \frac{\bar{\alpha}}{2} \right) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{8\bar{\alpha} \cos \bar{\phi}_m}{\pi^2 (2m-1)^2 - 4\bar{\alpha}^2} \right\}$ $\left[ \frac{(r/a)^{(2m-1)/2\alpha} (1-k)^{(2m-1)/2\alpha+2} - (a/r)^{(2m-1)/2\alpha} (k^2 - k^{(2m-1)/2\alpha})_k^{(2m-1)/2\alpha}}{1 - k^{(2m-1)/2\alpha}} - \frac{2\bar{\alpha}}{\pi (2m-1)} \left( \frac{r}{a} \right)^2 \right]$ $+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{8\bar{\alpha}^2 C_{(2m-1)/2\alpha}^{(a)} \cos \bar{\phi}_m \cosh(\kappa_{2m-1,n} + \zeta_{2m-1,n})}{\pi^2 (2m-1)^2 - 4\bar{\alpha}^2} \left[ \eta_{2m-1,n}^{-2} \right]^{(2m-1,n)} \cosh \kappa_{2m-1,n}$ $\left[ \frac{\xi_{2m-1,n} (1-k)^{(2m-1)/2\alpha+2} - q_{2m-1,n} (k^2 - k^{(2m-1)/2\alpha})_k^{(2m-1)/2\alpha}}{1 - k^{(2m-1)/2\alpha}} - \frac{\pi (2m-1)}{2\bar{\alpha}} g_{2m-1,n} \right]$ |  |                           |
| 3. Natural angular frequency:  |  |                           |
| $\omega_{2m-1,n}^2 = \frac{g}{a} \xi_{2m-1,n} \tanh \kappa_{2m-1,n}$ where $\xi_{2m-1,n}$ are roots of $C'_{(2m-1)/2\alpha}(\xi_{2m-1,n}) = 0$ .   |  |                           |

Table 4-49. Liquid Force and Moment Resultants

| Annular-Sector Tank  | Excitation: Harmonic Roll   |
|--|---|
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ .   |   |
| 1. $\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m_L a \omega^2 \phi_0 e^{i\omega t} \begin{pmatrix} \frac{2(1-k)^3}{1-k^2} \left( \frac{\sin \bar{\alpha}}{\sin \bar{\alpha} - (1+\cos \bar{\alpha})/3} \right) + \frac{16\bar{\alpha}}{\pi} \left( \frac{\sin \bar{\alpha}}{1+\cos \bar{\alpha}} \right) \sum_{m=1}^{\infty} \frac{1}{(2m-1)(1-k^2)[\pi^2(2m-1)^2 - 4\bar{\alpha}^2]} \cdot$ | $\left\{ \frac{-\bar{\alpha}^2}{\pi^2(2m-1)^2 - \bar{\alpha}^2} \left[ \frac{1 - 2k(2m-1)/2\alpha + 2}{1 - k(2m-1)/\alpha} + k \frac{(2m-1)/2\alpha + 1}{(2m-1)/\alpha + 3} + k^3 \frac{2\bar{\alpha}(1-k^3)}{\pi(2m-1)} \right] \right.$ |
| $\left. + \frac{\bar{\alpha}^2}{\pi^2(2m-1)^2 - \bar{\alpha}^2} \left[ \frac{[(2m-1)/2\alpha - 1](1-k(2m-1)/2\alpha + 1)(1-k(2m-1)/2\alpha + 2)}{[\pi^2(2m-1)^2 - \bar{\alpha}^2]} + \frac{(2m-1)/2\alpha - 1}{(k(2m-1)/2\alpha - k)(k-k(2m-1)/2)}$  | $-\frac{2\bar{\alpha}(1-k^3)}{3\pi(2m-1)} \left\{ + \frac{16\bar{\alpha}}{\pi} \left( \frac{\sin \bar{\alpha}}{1+\cos \bar{\alpha}} \right) \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \cdot \right.$  |
| $\left. \left[ \frac{[\bar{\alpha}^2/(\pi^2(2m-1)^2 - \bar{\alpha}^2)]}{(2m-1)(1-k^2)} \left( \eta_{2m-1,n}^2 - 1 \right) \left[ \pi^2(2m-1)^2 - 4\bar{\alpha}^2 \right] \kappa_{2m-1,n} \right\} \cdot \right.$   | $\left. \left[ \frac{\ell_{2m-1,n} (1-k(2m-1)/2\alpha + 2)}{1-k(2m-1)/\alpha} - \frac{q_{2m-1,n} (k-k(2m-1)/2\alpha)_k (2m-1)/2\alpha}{1-k(2m-1)/\alpha} - g_{2m-1,n} \left( \frac{2m-1}{4\alpha} \right) \right] \right\}$               |

Table 4-49. Liquid Force and Moment Resultants (continued)

|   |   |
|---|---|
| Annular-Sector Tank   | Excitation: Harmonic Roll   |
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$ (continued).   |   |
| $2. \begin{pmatrix} M_x \\ M_y \end{pmatrix} = m_L a^2 \omega^2 \phi_0 e^{i\omega t} \left\{ \frac{1-k^5}{5(h/a)(1-k^2)} \left( \frac{-[\sin \bar{\alpha} + 2(\cos \bar{\alpha} - 1)/\bar{\alpha}]}{2\sin \bar{\alpha}/\bar{\alpha} - (1 + \cos \bar{\alpha})} \right) + \frac{16\bar{\alpha}}{\pi(h/a)(1-k^2)} \left( \frac{\sin \bar{\alpha}}{1 + \cos \bar{\alpha}} \right) \sum_{m=1}^{\infty} \right.$ |   |
|   | $\frac{1}{(2m-1)[\pi^2(2m-1)^2 - 4\alpha^2]} \left\{ \frac{-\alpha^2}{\pi^2(2m-1)^2 - \alpha^2} \cdot \right.$  |
|   | $\left[ \frac{[(2m-1)/2\alpha - 3](1-k^{(2m-1)/2\alpha + 2})(1-k^{(2m-1)/2\alpha + 3})}{[(2m-1)/2\alpha - 3][(2m-1)/2\alpha + 3](1-k^{(2m-1)/\alpha})} + \frac{(2m-1)/2\alpha - k^3}{(k-k^2)^2} \frac{(2m-1)/2\alpha}{(2m-1)/2\alpha} \right]$  |
| $- \frac{2\bar{\alpha}(1-k^5)}{5\pi(2m-1)} \left\{ + \frac{4\bar{\alpha}}{\pi} \left( \frac{\sin \bar{\alpha}}{1 + \cos \bar{\alpha}} \right) \sum_{m=1}^{\infty} \frac{1}{(2m-1)[\pi^2(2m-1)^2 - 4\alpha^2]} (1-k^2)(\eta_{2m-1,n} - 1) \xi_{2m-1,n} \right.$  |   |
|   | $\frac{g_{2m-1,n}(1-k^{(2m-1)/2\alpha + 2}) - q_{2m-1,n}(k^2 - k^{(2m-1)/2\alpha})_k(2m-1)/2\alpha - g_{2m-1,n}(2m-1)/4\alpha(1-k^{(2m-1)/\alpha})}{1-k^{(2m-1)/\alpha}}$   |
|   | $\left\{ \left[ \tanh \kappa_{2m-1,n} + \frac{2}{\kappa_{2m-1,n}} \left( \frac{1}{\cosh \kappa_{2m-1,n}} - 1 \right) \right] \left[ \frac{-\alpha^2}{\pi^2(2m-1)^2 - \alpha^2} \left( \frac{2}{\pi \xi_{2m-1,n}} - k \xi_{2m-1,n} \right) \right] \right.$  |
|   | $+ N_0(\xi_{2m-1,n}) \left. \right\} + \frac{2\bar{\alpha}^2 \xi_{2m-1,n}^2 N_2(\xi_{2m-1,n})}{[\pi^2(2m-1)^2 - \alpha^2] \kappa_{2m-1,n} \cosh \kappa_{2m-1,n}} \left\{ + \frac{2m_L g \alpha (1-k^3)}{3\bar{\alpha}(1-k^2)} \left( \frac{\sin \bar{\alpha}}{1 - \cos \bar{\alpha}} \right) \right.$ |

Table 4-49. Liquid Force and Moment Resultants (continued)

| Annular-Sector Tank   | Excitation: Harmonic Roll |
|---|---------------------------|
| $3. \quad M_z = m_L a^2 \omega^2 \phi_0 e^{i\omega t} \left\{ \frac{1+k^2}{2} + \frac{32\bar{\alpha}}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)(1-k^2) [\pi^2 (2m-1)^2 - 4\bar{\alpha}^2]} \right. \\ - \frac{[(2m-1)/2\alpha-2](1-k^2)}{[(2m-1)/2\alpha-2] [(2m-1)/2\alpha+2] (1-k^2)} \frac{(2m-1)/2\alpha+2}{(2m-1)/\alpha} \\ + \frac{32\bar{\alpha}}{\pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{N_1(\xi_{2m-1,n}) \tanh \kappa_{2m-1,n}}{(2m-1) [\pi^2 (2m-1)^2 - 4\bar{\alpha}^2] (\eta_{2m-1,n}^2 - 1) \kappa_{2m-1,n}} \\ \left. \left[ \frac{q_{2m-1,n} (1-k^2) (2m-1)/2\alpha+2}{1-k} - \frac{q_{2m-1,n} (k^2 - k^2 (2m-1)/2\alpha) k (2m-1)/2\alpha}{1-k} \right] \right\} \\ - 8_{2m-1,n} \left( \frac{2m-1}{4\alpha} \right) \left. \right\}$ |                           |



## IV. Rigid Tanks

## 4.8 Rectangular Tank

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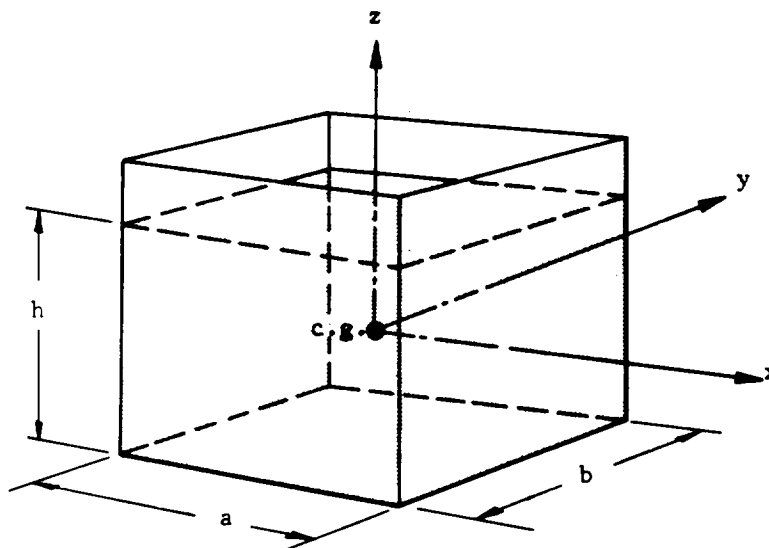
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## IV. Rigid Tanks

## 4.8 Rectangular Tank



Container: The tank is rectangular having base dimensions  $a$  and  $b$  in the  $x$  and  $y$  directions, respectively, and is filled with a liquid to a depth  $h$ .

Coordinate System: The origin is located at the center of gravity of the undisturbed fluid.

References: (76)



Table 4-50. Boundary Conditions, Velocity Potential, and Natural Frequency

| Rectangular Tank   | Excitation: Harmonic Translation                                      |
|--|---|
| Translation in the x-direction, $x = x_0 \sin \omega t$ .  |   |
| 1. Boundary conditions:  |   |
| (a) $\left( \frac{\partial \phi}{\partial x} \right)_{x=\pm a/2} = -\omega x_0 \cos \omega t$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=\pm h/2} = 0$ |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=\pm h/2} = 0$   |   |
| 2. Velocity potential:   |   |
| $\phi = -\omega x_0 \cos \omega t \left[ x + \sum_{n=0}^{\infty} \frac{4a(-1)^n \sin[(2n+1)(\pi/a)x] \cosh[(2n+1)(\pi/a)(z+h/2)]}{\pi^2 (2n+1)^2 (\eta_n^2 - 1) \cosh[(2n+1)(\pi h)/a]} \right]$ |   |
| 3. Natural angular frequency:  |   |
| $\omega_n^2 = g(2n+1)(\pi/a) \tanh \left[ (2n+1) \frac{\pi h}{a} \right]$  |   |



Table 4-50. Boundary Conditions, Velocity Potential, and Natural Frequency (continued)

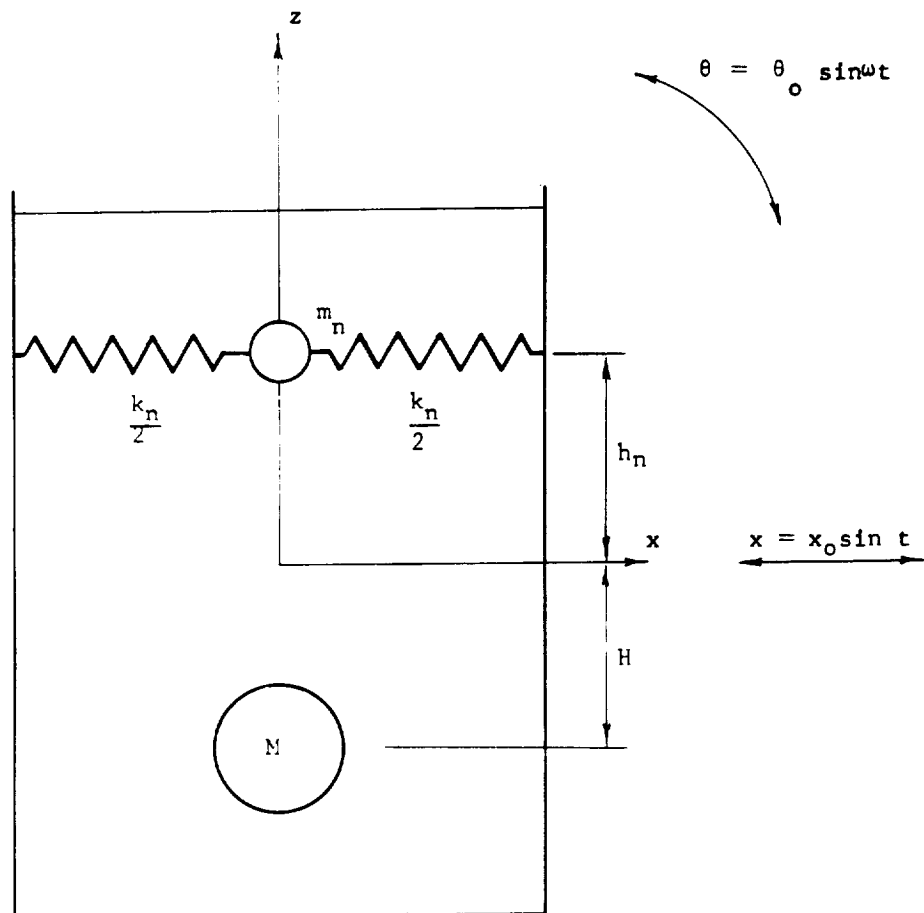
| Rectangular Tank   | Excitation: Harmonic Pitching  |
|--|--|
| Pitching about the y-axis, $\theta = \theta_0 \sin \omega t$ .   |  |
| 1. Boundary conditions:  |  |
| (a) $\left( \frac{\partial \phi}{\partial x} \right)_{x=-a/2} = -\omega z \theta_0 \cos \omega t$  | (b) $\left( \frac{\partial \phi}{\partial z} \right)_{z=-h/2} = \omega x \theta_0 \cos \omega t$ |
| (c) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=h/2} = 0$   |  |
| 2. Velocity potential:   |  |
| $\phi = -\omega \theta_0 \cos \omega t \left\{ \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi^3 (2n+1)^3} \left[ \frac{h^2 \sin[(2n+1)(\pi/h)x] \sinh[(2n+1)(\pi/h)z]}{\cosh[(2n+1)(\pi a)/2h]} \right] \right.$ $+ \frac{a^2 \sin[(2n+1)(\pi/a)x] \cosh[(2n+1)(\pi/a)(z-h/2)]}{\sinh[(2n+1)(\pi/a)h]} \left. \right\} + \sum_{n=0}^{\infty} \frac{4a(-1)^n \sin[(2n+1)(\pi/a)x] \cosh[(2n+1)(\pi/a)(z+h/2)]}{\pi^2 (2n+1)^2 (\eta_n^2 - 1) \cosh[(2n+1)(\pi h)/a]}$ $\left[ \frac{h}{2} - \frac{2a \tanh[(2n+1)(\pi h)/2a]}{\pi (2n+1)} + \frac{g}{\omega_n^2} \right]$ |  |
| 3. Natural angular frequency:  |  |
| $\omega_n^2 = g (2n+1)(\pi/a) \tanh \left[ (2n+1) \frac{\pi h}{a} \right]$   |  |

Table 4-51. Liquid Force and Moment Resultants

| Rectangular Tank   |  | Excitation: Harmonic Translation and/or Pitching |
|--|--|--|
| Translation in the x-direction, $x = x_0 \sin \omega t$ .      |  |  |
| 1.   | $F_x = m_L \omega^2 x_0 \sin \omega t \left[ 1 + \sum_{n=0}^{\infty} \frac{8 \tanh[(2n+1)\pi r_1]}{3\pi^3 r_1 (2n+1)^3 (\eta_n^2 - 1)} \right]$  |  |
| 2.   | $M_y = m_L h \omega^2 x_0 \sin \omega t \left[ \frac{1}{12r_1^2} + \sum_{n=0}^{\infty} \frac{8 \tanh[(2n+1)\pi r_1]}{3\pi^3 r_1 (2n+1)^3 (\eta_n^2 - 1)} \left( \frac{1}{2} - \frac{\tanh[(2n+1)\pi r_1/2]}{(2n+1)\pi r_1/2} + \frac{g}{h\omega_n^2} \right) \right]$  |  |
| Pitching about the y-axis, $\theta = \theta_0 \sin \omega t$ . |  |  |
| 1.   | $F_x = m_L h \omega^2 \theta_0 \sin \omega t \left[ \frac{1}{12r_1^2} + \sum_{n=0}^{\infty} \frac{8 \tanh[(2n+1)\pi r_1]}{3\pi^3 r_1 (2n+1)^3 (\eta_n^2 - 1)} \left( \frac{1}{2} - \frac{\tanh[(2n+1)\pi r_1/2]}{(2n+1)\pi r_1/2} + \frac{g}{h\omega_n^2} \right) \right]$   |  |
| 2.   | $M_y = m_L h^2 \omega^2 \theta_0 \sin \omega t \left[ \frac{\bar{I}}{m_L h^2} + 2 \sum_{n=0}^{\infty} \frac{8g \tanh[(2n+1)\pi r_1]}{3\pi^3 r_1 (2n+1)^3 h\omega_n^2} \left( \frac{1}{2} - \frac{\tanh[(2n+1)\pi r_1/2]}{(2n+1)\pi r_1/2} + \frac{g}{2h\omega_n^2} \right) + \sum_{n=0}^{\infty} \frac{8 \tanh[(2n+1)\pi r_1]}{3\pi^3 r_1 (2n+1)^3 (\eta_n^2 - 1)} \left( \frac{1}{2} - \frac{\tanh[(2n+1)\pi r_1/2]}{(2n+1)\pi r_1/2} + \frac{g}{h\omega_n^2} \right)^2 + \frac{\theta_0 \sin \omega t}{12r_1^2} \right]$ |  |

Table 4-52. Model Analysis

| Rectangular Tank   | Spring-Mass Model |
|--|-------------------|
| Excitation: Harmonic Translation and/or Pitching   |                   |
| <p>Figure 4-43 shows a diagram of the spring-mass model used in representing the dynamic response of a liquid in a rectangular tank when subjected to harmonic translation in the x-direction and/or pitching about the y-axis.</p> <p><u>Coordinate System:</u></p> <p>The origin is located at the center of gravity of the undisturbed liquid.</p> <p><u>Model Description:</u></p> <p>The components of the system are as follows:</p> <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the z-axis at a distance <math>H</math> below the coordinate origin.</li> <li>2. A set of movable masses <math>m_n</math> is distributed along the z-axis when the container is at rest at distances <math>h_n</math> above the origin. These modal masses are constrained by springs <math>n</math> having spring stiffness coefficients <math>k_n</math> to remain in the xy-plane and to move only in a direction parallel to the x-axis.</li> </ol> <p><u>Equations of Motion:</u></p> <p>The equations, obtained from either an equilibrium or energy formulation, are as follows:</p> <ol style="list-style-type: none"> <li>1. Slosh Mass Equation for Translation: <math display="block">m_n \ddot{x} - k_n x = m_n x_o \omega^2 \sin \omega t</math> </li> <li>2. Slosh Mass Equation for Pitching: <math display="block">m_n \ddot{x} + k_n x = (m_n g + h_n k_n) \theta_o \sin \omega t</math> </li> </ol> <p>From these equations, the model force in the x-direction and the moment about the y-axis can be found (see Table 4-53).</p> |                   |



Rectangular Tank

Spring-Mass Model

Excitation: Harmonic Translation and/or Pitching

Figure 4-43. Equivalent Mechanical Model



Table 4-53. Model Force and Moment Resultants

| Rectangular Tank  | Spring-Mass Model   |
|---|---|
| Excitation: Harmonic Translation and/or Pitching                  |   |
| 1. Translation in the x-direction, $x = x_o \sin \omega t$ .      |   |
| (a)   | $F_x = m_L \omega^2 x_o \sin \omega t \left[ \frac{M}{m_L} + \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{\eta_n^2}{2 - 1} \right) \right]$   |
| (b)   | $M_y = m_L h \omega^2 x_o \sin \omega t \left[ \frac{Mh}{m_L h} + \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{h}{h} + \frac{g}{h \omega_n^2} \right) \left( \frac{\eta_n^2}{2 - 1} \right) \right]$  |
| 2. Pitching about the y-axis, $\theta = \theta_o \sin \omega t$ . |   |
| (a)   | $F_x = m_L h \omega^2 \theta_o \sin \omega t \left[ \frac{Mh}{m_L h} + \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{h}{h} + \frac{g}{h \omega_n^2} \right) \left( \frac{\eta_n^2}{2 - 1} \right) \right]$   |
| (b)   | $M_y = m_L h^2 \omega^2 \theta_o \sin \omega t \left[ \frac{I}{m_L h^2} + \frac{M}{m_L} \left( \frac{h}{h} \right)^2 + \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{h}{h} + \frac{g}{h \omega_n^2} \right) \left( \frac{\eta_n^2}{2 - 1} \right) \right]$ |
|   | $+ \theta_o \sin \omega t \left[ \frac{Mh}{m_L h} + \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{h}{h} + \frac{g}{h \omega_n^2} \right) \right]$  |



Table 4-54. Model Elements

| Rectangular Tank                                 | Spring-Mass Model  |
|--|--|
| Excitation: Harmonic Translation and/or Pitching |  |
| Natural Frequency                                | $\omega_n^2 = \frac{g}{a} (2n+1)\pi \tanh \left[ (2n+1)\pi r_1 \right]$  |
| Spring Constant                                  | $k_n = m_n \omega_n^2$ (fig. 4-44)   |
| Ratio of Spring Constant to Fluid Weight         | $\frac{hk_n}{m_L g} = \frac{8 \tanh^2 \left[ (2n+1)\pi r_1 \right]}{\left[ (2n+1)\pi \right]^2}$   |
| Ratio of Slosh Mass to Fluid Mass                | $\frac{m_n}{m_L} = \frac{8 \tanh \left[ (2n+1)\pi r_1 \right]}{\left[ (2n+1)\pi \right]^3 r_1}$ (fig. 4-46)  |
| Ratio of Fixed Mass to Fluid Mass                | $\frac{M}{m_L} = 1 - \sum_{n=0}^{\infty} \frac{m_n}{m_L}$ (fig. 4-46)  |
| Ratio of Slosh Mass Coordinate to Fluid Depth    | $\left  \frac{h_n}{h} \right  = \frac{1}{2} - \frac{\tanh \left[ (2n+1)\pi r_1 / 2 \right]}{(2n+1)\pi r_1 / 2}$ (fig. 4-45)  |
| Ratio of Fixed Mass Coordinate to Fluid Depth    | $\left  \frac{H}{h} \right  = - \frac{m_L}{M} \sum_{n=0}^{\infty} \left( \frac{m_n}{m_L} \right) \left( \frac{h_n}{h} \right)$ (fig. 4-45)   |
| Moment of Inertia of Solidified Fluid            | $\frac{I_s}{m_L h^2} = \frac{1}{12} \left[ \left( \frac{a}{h} \right)^2 + 1 \right]$   |
| Effective Moment of Inertia of Fluid             | $\frac{\bar{I}}{m_L h^2} = \frac{I_s}{m_L h^2} \left[ 1 - \frac{4}{r_1^2 + 1} + \frac{768}{(r_1^2 + 1)^5 r_1} \cdot \sum_{n=0}^{\infty} \frac{\tanh \left[ (2n+1)\pi r_1 / 2 \right]}{(2n+1)^5} \right]$ (fig. 4-47) |
| Moment of Inertia of Fixed Mass                  | $\frac{I}{m_L h^2} = \frac{\bar{I}}{m_L h^2} - \frac{M}{m_L} \left( \frac{H}{h} \right)^2 - \sum_{n=0}^{\infty} \frac{m_n}{m_L} \left( \frac{h_n}{h} \right)^2$ (fig. 4-47)  |



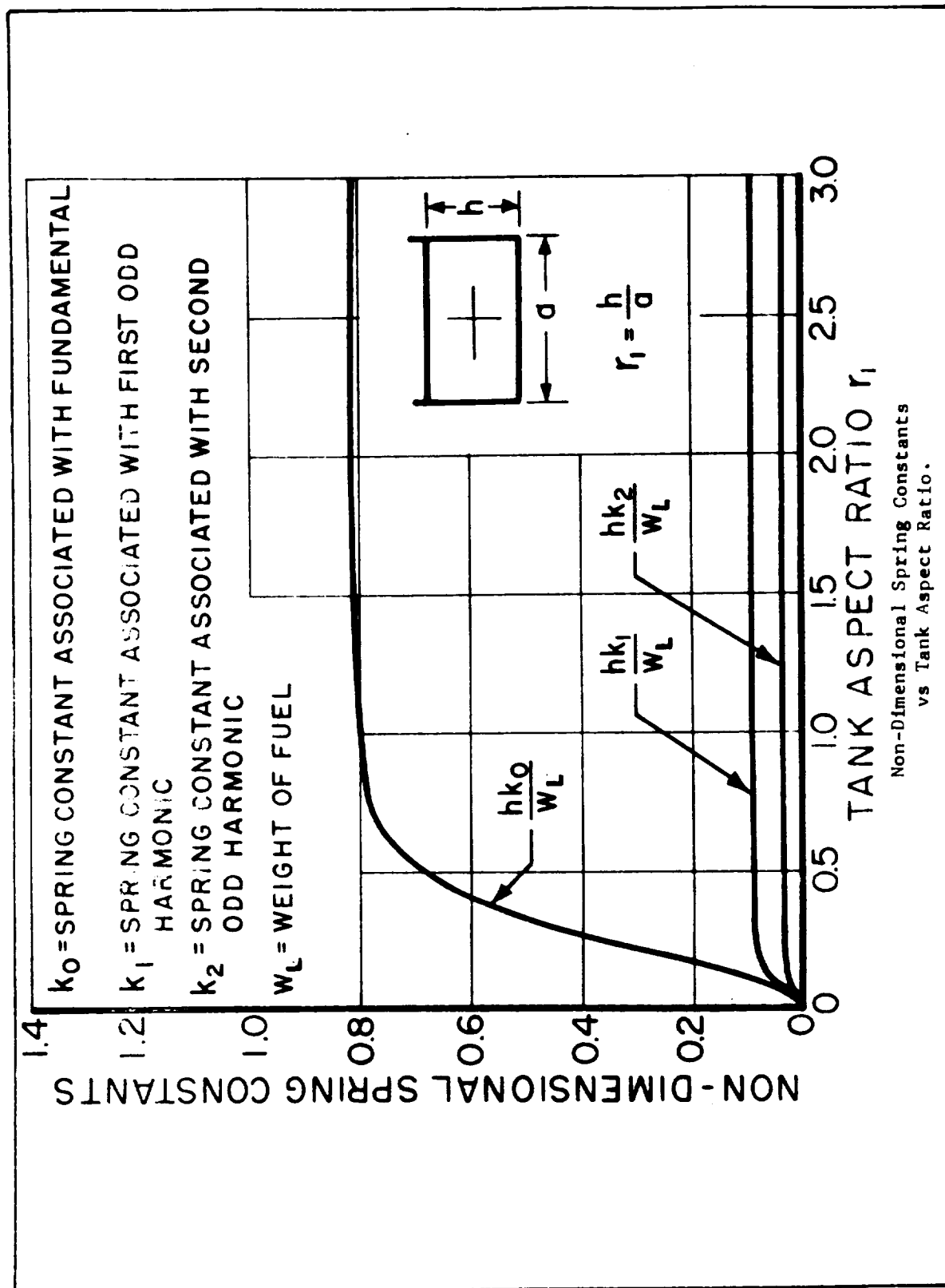


Figure 4-44. Model Element Graph



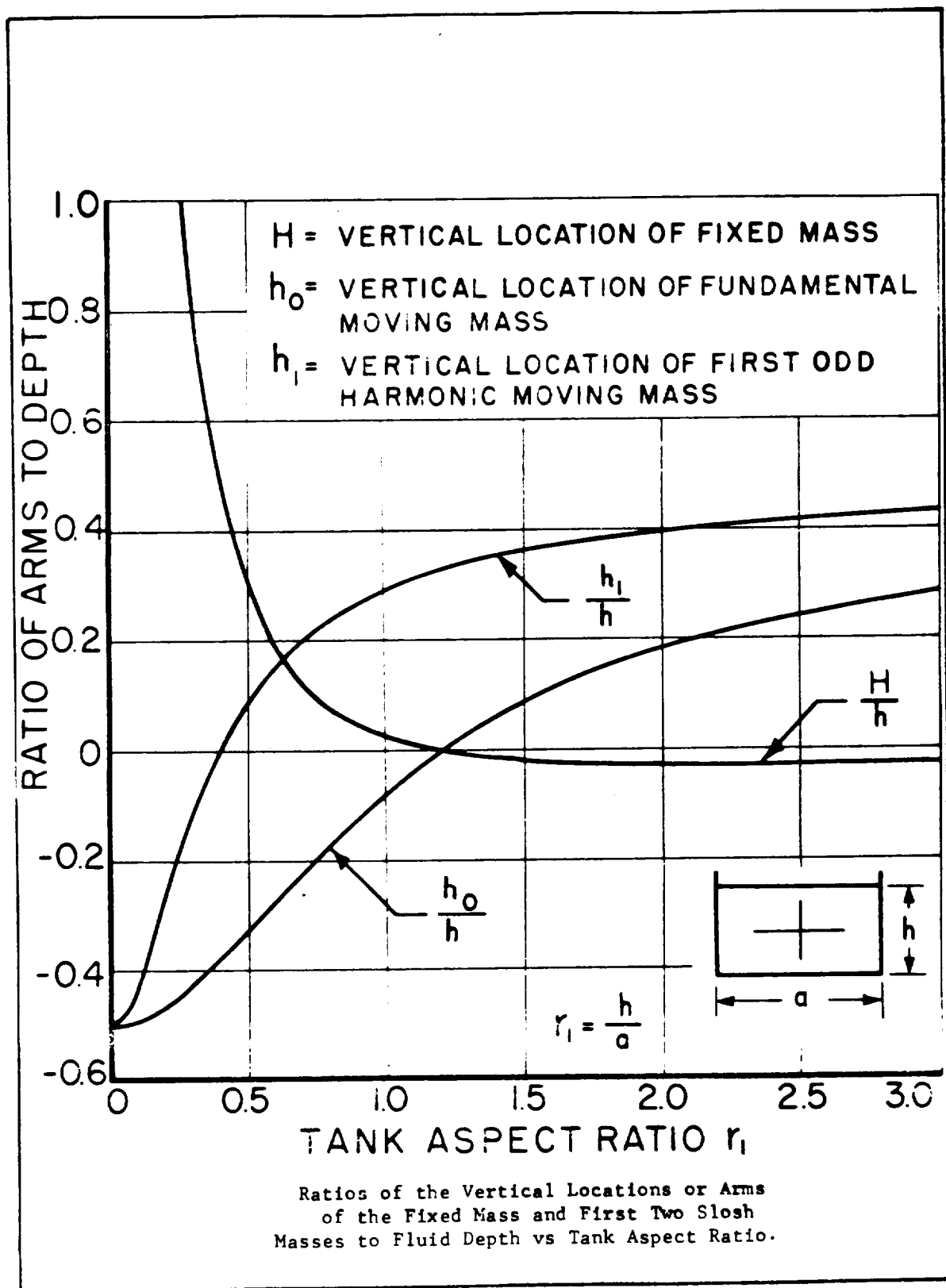


Figure 4-45. Model Element Graph





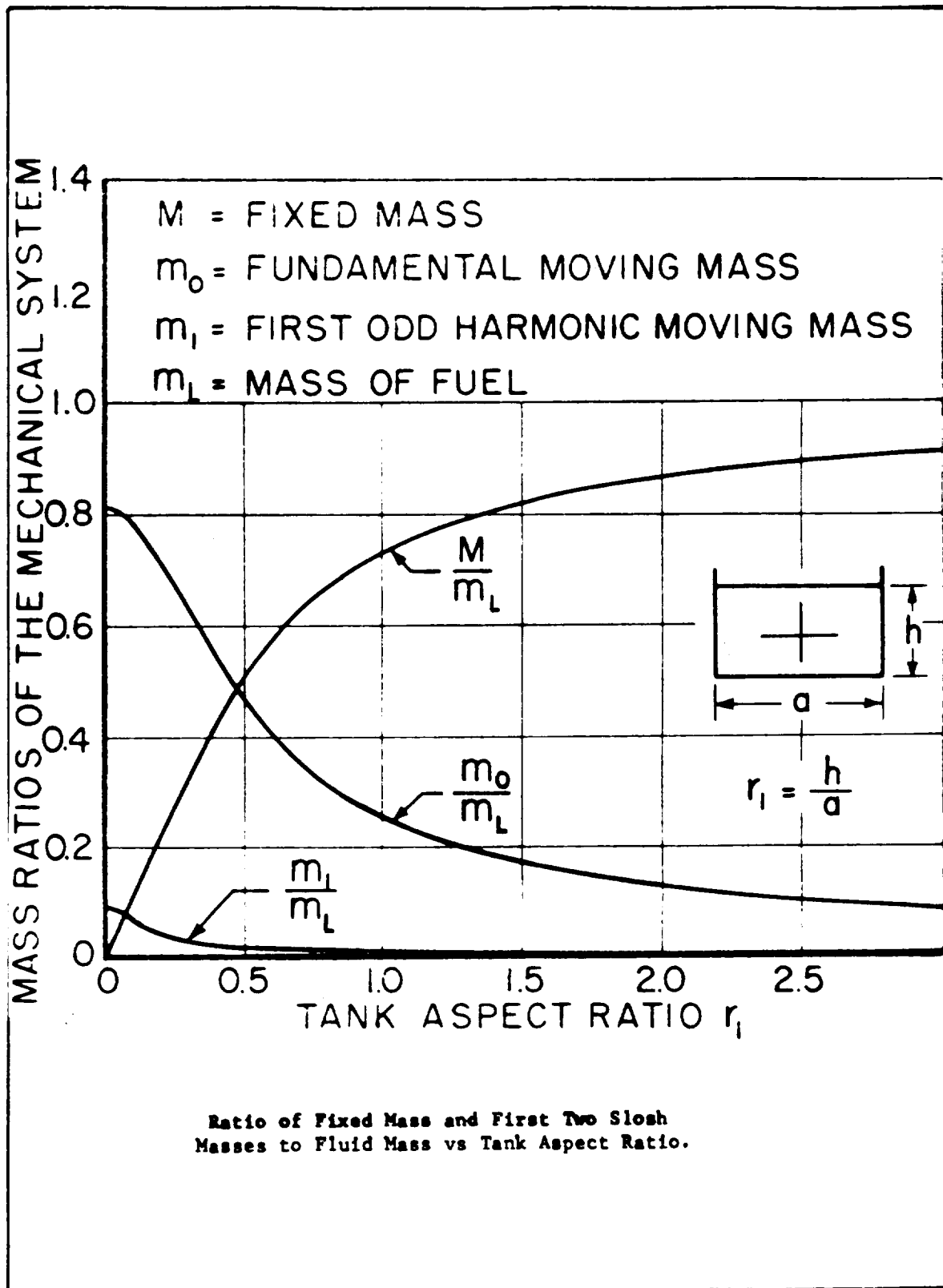
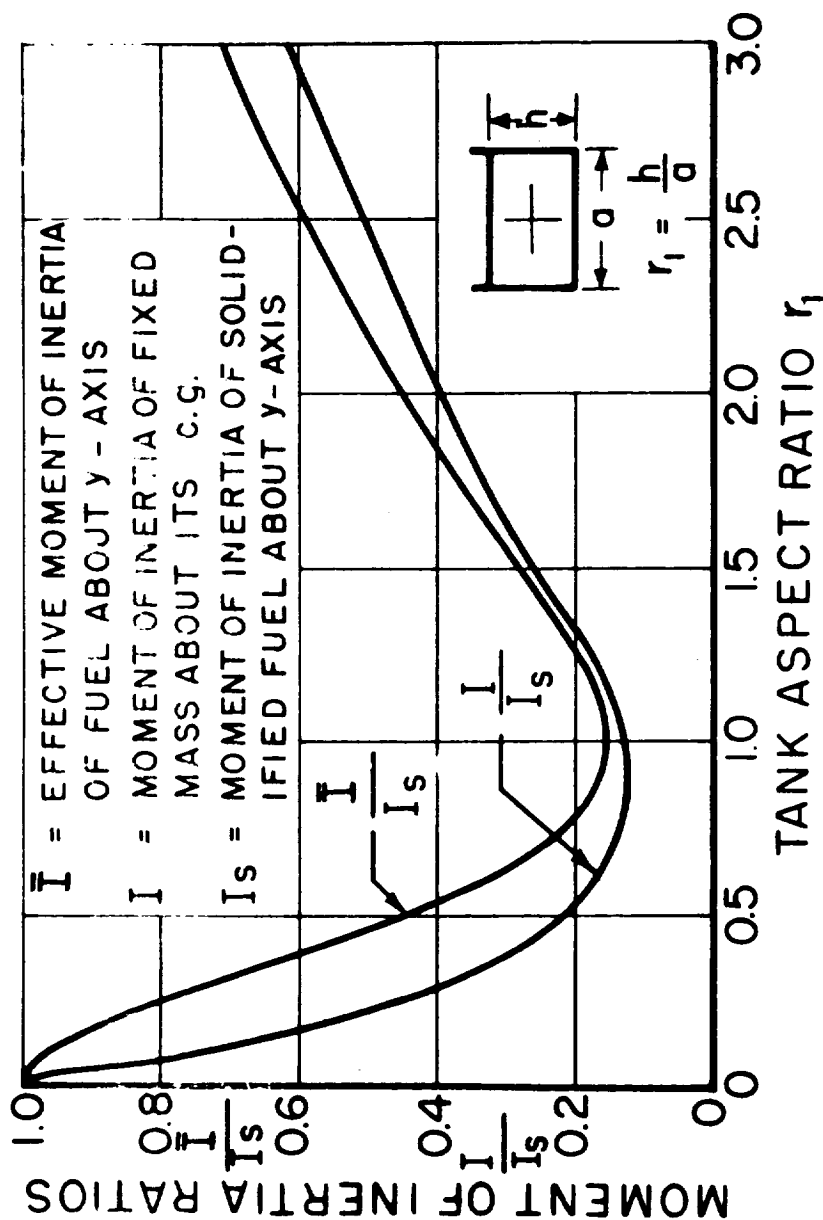


Figure 4-46. Model Element Graph





Ratios of Moments of Inertia of Fluid and Fixed Mass to the Moment of Inertia of the Solidified Fluid vs Tank Aspect Ratio.

Figure 4-47. Model Element Graph



Table 4-55. Boundary Conditions, Velocity Potential, and Natural Frequency

| Rectangular Tank   | Excitation: Harmonic Roll  |
|--|--|
| Roll about the z-axis, $\phi = \phi_0 \sin \omega t$ .   |  |
| 1. Boundary conditions:  |  |
| (a) $\left( \frac{\partial \phi}{\partial x} \right)_{x=\pm a/2} = \omega y \phi_0 \cos \omega t$  | (b) $\left( \frac{\partial \phi}{\partial y} \right)_{y=\pm b/2} = -\omega x \phi_0 \cos \omega t$             |
| (c) $\left( \frac{\partial \phi}{\partial z} \right)_{z=\pm h/2} = 0$  | (d) $\left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right)_{z=\pm h/2} = 0$ |
| 2. Velocity potential:   |  |
| $\phi = \omega \phi_0 \cos \omega t \left\{ \sum_{n=0}^{\infty} \frac{4(-1)^n}{3 \pi (2n+1)^3} \left[ \frac{b^2 \sin[(2n+1)(\pi/b)y] \sinh[(2n+1)(\pi/b)x]}{\cosh[(2n+1)(\pi a)/2b]} \right] \right.$ $- \frac{a^2 \sin[(2n+1)(\pi/a)x] \sinh[(2n+1)(\pi/a)y]}{\cosh[(2n+1)\pi b/2a]} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{16ab(-1)^{m+n} [b^2(2n+1)^2 - a^2(2m+1)^2]}{\pi^4 (2m+1)^2 (2n+1)^2 [b^2(2n+1)^2 + a^2(2m+1)^2]} (\eta^2 - 1)_{mn} \cdot$ $\left. \sin[(2m+1)\frac{\pi}{b}y] \sin[(2n+1)\frac{\pi}{a}x] \operatorname{sech} \left[ \frac{\pi h}{ab} \sqrt{b^2(2n+1)^2 + a^2(2m+1)^2} \right] \right\}$ $\cosh \left[ \frac{\pi}{ab} \left( z + \frac{h}{2} \right) \sqrt{b^2(2n+1)^2 + a^2(2m+1)^2} \right] \Bigg\}$ |  |
| 3. Natural angular frequency:  |  |
| $\omega_{mn}^2 = g \frac{\pi}{ab} \sqrt{b^2(2n+1)^2 + a^2(2m+1)^2} \quad \tanh \left[ \frac{\pi h}{ab} \sqrt{b^2(2n+1)^2 + a^2(2m+1)^2} \right]$   |  |

Table 4-56. Liquid Moment Resultant

| Rectangular Tank   | Excitation: Harmonic Roll |
|--|---------------------------|
| Roll about the z-axis, $\phi = \phi_0 \sin \omega t$ .   |                           |
| $M_z = I_s r_1^2 \omega^2 \theta_0 \sin \omega t \left[ \frac{\bar{I}}{I_s} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{768 [r_3^2 (2m+1)^2 - (2n+1)^2]^2 \tanh[\pi r_1 \sqrt{r_3^2 (2m+1)^2 + (2n+1)^2}]}{\pi r_1 (1+r_3^2)(2m+1)^4 (2n+1)^4 [r_3^2 (2m+1)^2 + (2n+1)^2]^{3/2} (r_{mn}^2 - 1)} \right]$ |                           |

Table 4-57. Model Analysis

| Rectangular Tank  | Torsional Spring-Mass Model |
|---|-----------------------------|
| Excitation: Harmonic Roll   |                             |
| <p>Figure 4-48 shows a diagram of the torsional spring-mass model used in representing the dynamic response of a liquid in a rectangular tank when subjected to harmonic roll about the z-axis.</p>   |                             |
| <p><u>Coordinate System:</u></p> <p>The origin is located at the center of gravity of the undisturbed liquid.</p>   |                             |
| <p><u>Model Description:</u></p> <p>The components of the system are as follows:</p> <ol style="list-style-type: none"> <li>1. A fixed mass <math>M</math> having a moment of inertia <math>I</math> is rigidly connected to the tank and is located on the z-axis at a distance <math>H</math> below the coordinate origin.</li> <li>2. A set of movable masses <math>m_{mn}</math> is constrained by torsional springs having stiffness coefficients <math>k_{mn}</math> to remain in planes parallel to the xy-plane and at distances <math>h_{mn}</math> above it.</li> </ol> |                             |
| <p><u>Equation of Motion:</u></p> <p>The equation, obtained from either an equilibrium or energy formulation, is as follows:</p>  |                             |
| $I_{mn} \ddot{\phi} - k_{mn} \phi = k_{mn} \phi_o \sin \omega t$  |                             |
| <p>From this equation, the moment about the z-axis can be found (see Table 4-58).</p>   |                             |

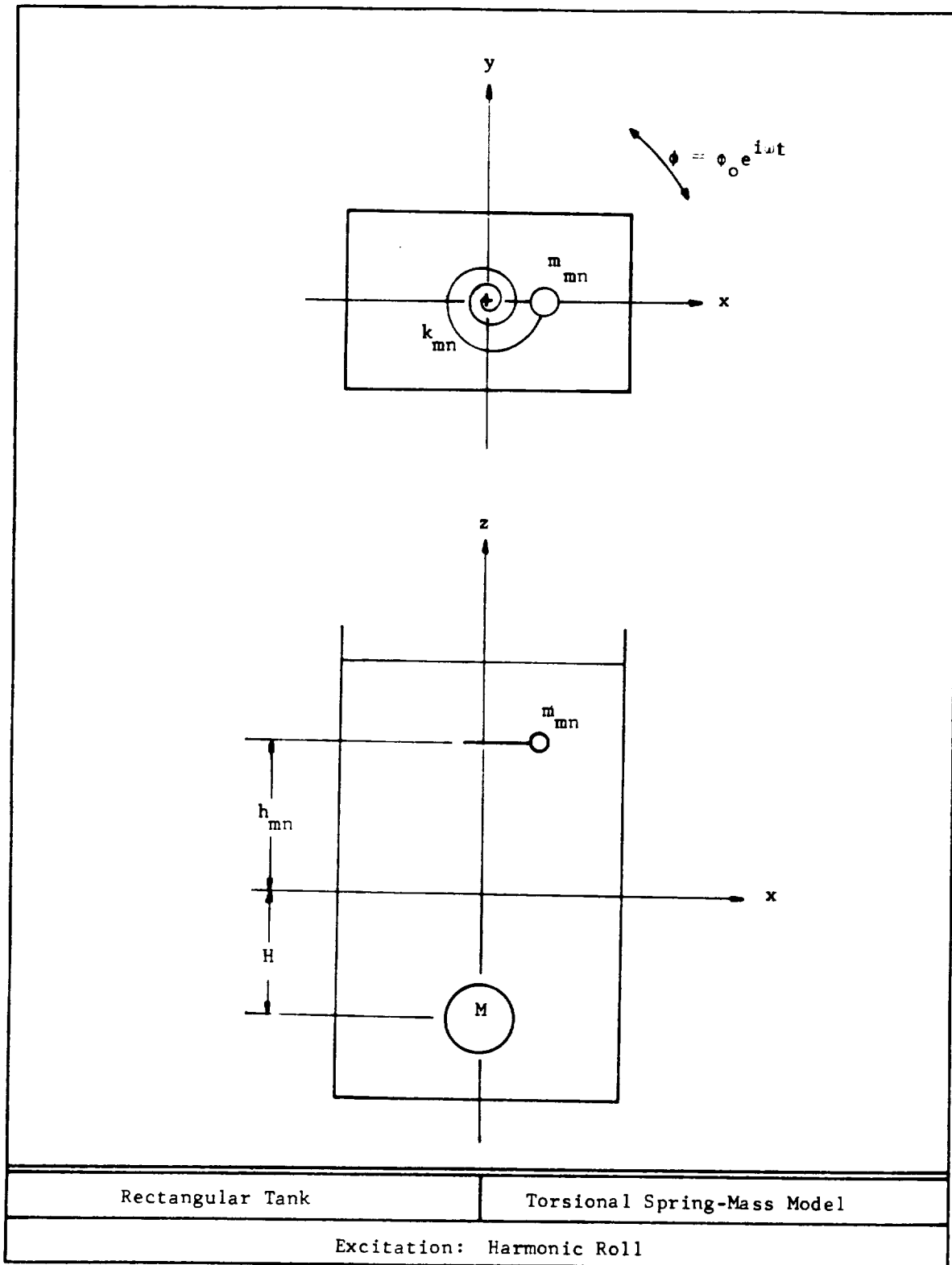


Figure 4-48. Equivalent Mechanical Model



Table 4-58. Model Moment Resultant

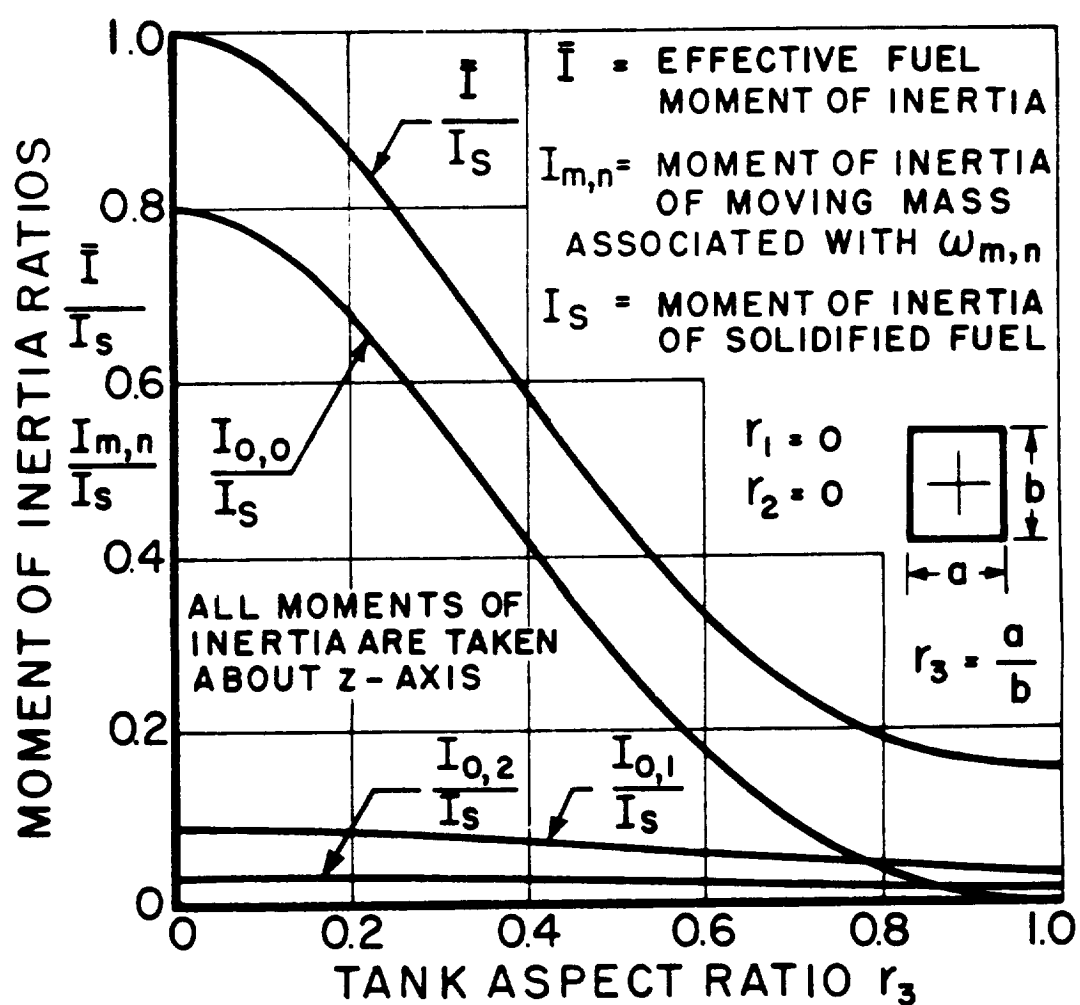
| Rectangular Tank   | Torsional Spring-Mass Model |
|--|-----------------------------|
| Excitation: Harmonic Roll  |                             |
| Roll about the z-axis, $\phi = \phi_0 \sin \omega t$ .   |                             |
| $M_z = I_s r_1^2 \omega^2 \phi_0 \sin \omega t \left[ \frac{I}{I_s} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{mn}}{I_s} \left( \frac{\eta_{mn}^2}{\eta_{mn}^2 - 1} \right) \right]$ |                             |



Table 4-59. Model Elements

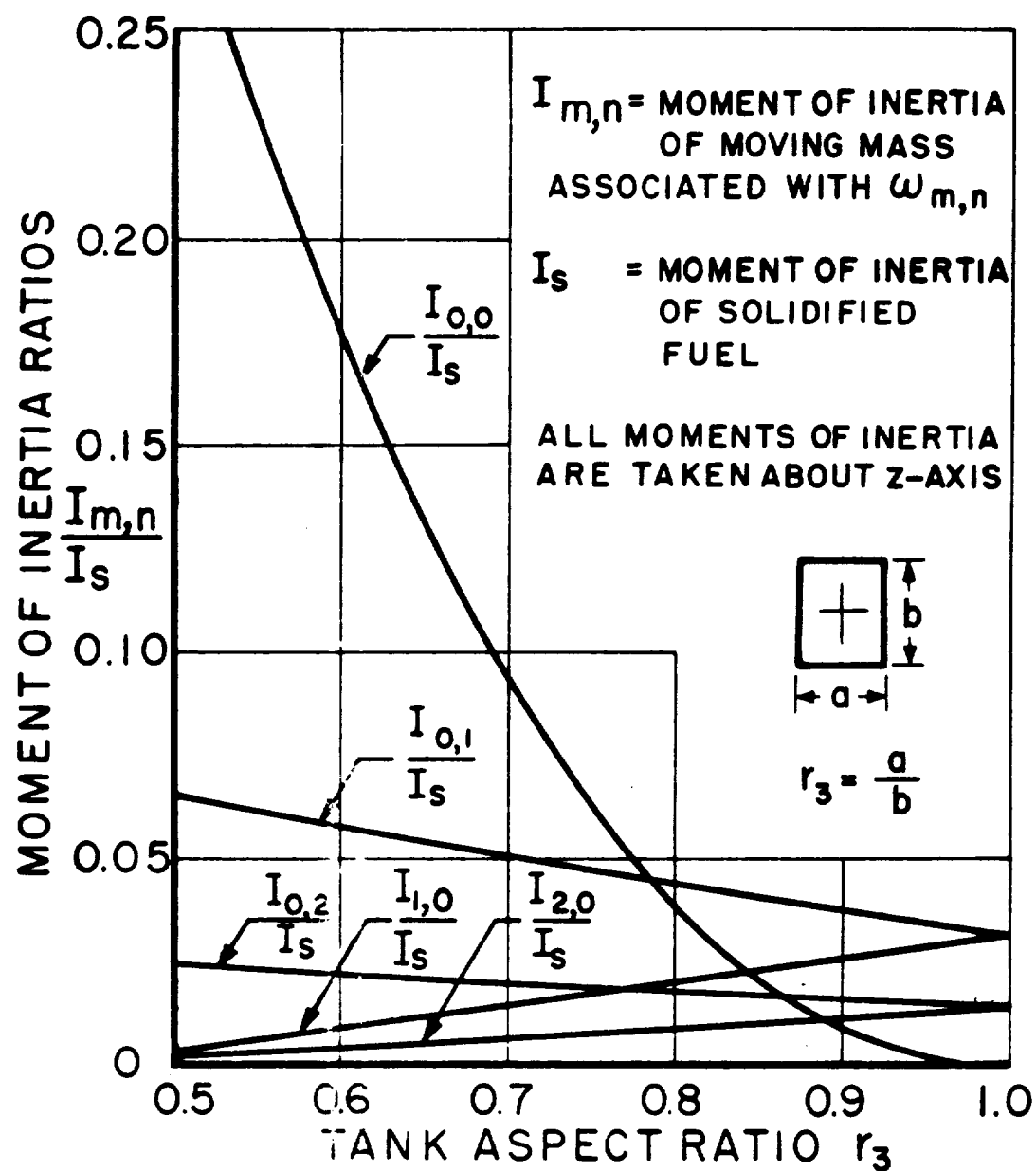
| Rectangular Tank                      | Torsional Spring-Mass Model  |
|---------------------------------------|--|
| Excitation: Harmonic Roll             |  |
| Natural Frequency                     | $\omega_{mn}^2 = \frac{\pi g r_1}{h} \sqrt{r_3^2 (2m+1)^2 + (2n+1)^2} \cdot \tanh \left[ \pi r_1 \sqrt{r_3^2 (2m+1)^2 + (2n+1)^2} \right]$   |
| Torsional Spring Constant             | $k_{mn} = I_{mn} \omega_{mn}^2$  |
| Moment of Inertia of Solidified Fluid | $I_s = \frac{m}{12} a^2 + b^2$   |
| Effective Moment of Inertia of Fluid  | $\frac{\bar{I}}{I_s} = 1 - \frac{4}{r_3^2 + 1} + \frac{768}{\pi^5 r_3 (r_3^2 + 1)} \sum_{n=0}^{\infty} \frac{\tanh \left[ (2n+1) \pi r_3 / 2 \right]}{(2n+1)^5} \quad (\text{fig. 4-49, 4-50})$  |
| Moment of Inertia of Slosh Mass       | $\frac{I_{mn}}{I_s} = \frac{768 \left[ r_3^2 (2m+1)^2 - (2n+1)^2 \right]^2}{\pi^6 (1+r_3^2) (2m+1)^4 (2n+1)^4 \left[ r_3^2 (2m+1)^2 + (2n+1)^2 \right]} \cdot \frac{\tanh \left[ \pi r_1 \sqrt{r_3^2 (2m+1)^2 + (2n+1)^2} \right]}{\pi r_1 \sqrt{r_3^2 (2m+1)^2 + (2n+1)^2}} \quad (\text{fig. 4-49, 4-50})$ |
| Moment of Inertia of Fixed Mass       | $\frac{I}{I_s} = \frac{\bar{I}}{I_s} - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{mn}}{I_s}$   |





Roll Moment of Inertia Ratios for Shallow Tank vs Tank Aspect Ratio.

Figure 4-49. Model Element Graph



Roll Moment of Inertia Ratios for Shallow Tank vs Tank Aspect Ratio.

Figure 4-50. Model Element Graph

## V. Propellant SLOSH Suppression

## 5.1 Anti-SLOSH Devices

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## V. Propellant Slosh Suppression

### 5.1 Anti-Slosh Devices

5.1.1 Introduction. Sloshing becomes detrimental when the wave amplitudes of tanked propellants, agitated by oscillations, approach breaking height. This sloshing of the liquid exerts considerable forces on the missile structure, affects the controls and renders any liquid level measuring device inaccurate. This means that a sloshing damping device has to be provided. The term damping usually refers to the checking of a motion due to resistance, as by friction or any other similar cause. It is of especial significance in connection with the diminishing amplitude of an oscillation, as that of a liquid swinging inside a moving container. Unless energy is supplied during each cycle, the amplitude of the fluid motion falls off at each successive oscillation by an amount commonly expressed in terms of the decrement, or damping factor, which is the ratio of any one amplitude to that next succeeding it in the same sense and direction.

Many of the force and moment resultants given in Chapter IV require a damping factor before they can be evaluated. The accuracy of these equations and consequently their usefulness is therefore very much dependent on the value chosen for the damping parameter. When a large propellant tank has no damping devices the resistance to fluid motion at the container walls is generally accepted to be nil. That is, the fluid damping is negligible. However, when damping systems are introduced, the resulting damping factor depends on many variables and is thereby extremely difficult to obtain. For some damping devices, such as fixed ring baffles, a sound analytical approach can be effectively used while in most cases an experimental approach is the only way of obtaining a meaningful damping factor.



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.2 Design Requirements of an Anti-Slosh Device. Studies (40), (42) show that the depth below the undisturbed free surface at which the liquid no longer participates in surface oscillations is approximately equal to one-fourth the tank diameter. Therefore a damping device need only extend to such a depth. The liquid level, however, changes as the container empties thereby requiring either a damping device capable of moving down at a rate equal to the draining rate or a damping device which is fixed to the wall but extends to the bottom of the container. The first stipulation suggests some type of floating device, whereas the second suggests either longitudinal partitions or an evenly spaced baffle system covering the inner periphery of the tank and distributed throughout the depth of the liquid.

The amount of damping needed to stabilize a missile is determined from analog or digital computer studies of the over-all dynamic characteristics of the vehicle. In some cases, stability studies might indicate no need for anti-slosh baffles in a particular tank. Generally, however, the designer is faced with the problem of optimizing a damping system by integrating into his design the requirements that follow:

1. Produce a high damping effect.
2. Pay a minimum weight penalty.
3. Utilize minimum space.
4. Absorb the liquid forces and moments or transfer them uniformly to the tank structure, thereby avoiding points of high stress concentration.

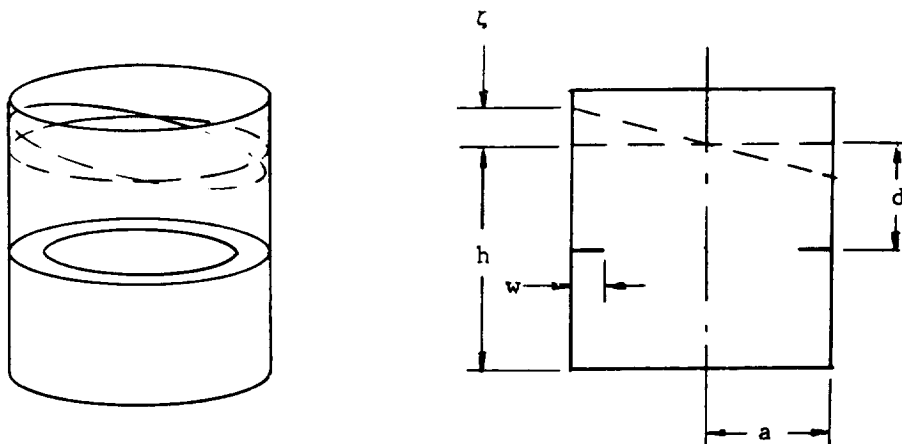
## V. Propellant Slopsh Suppression

## 5.1 Anti-Slopsh Devices (continued)

5. Function throughout environmental changes, such as temperature.
6. Not interfere with other operations, such as the emptying of the container or the measurement of the liquid level.
7. Be easy to assemble.
8. Not cause damage to the tank or other built-in equipment during transportation of the missile.
9. Not interfere with cleaning operations.

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices

Ring Baffles

where  $\zeta$  = amplitude of liquid measured at the tank wall

$h$  = depth of liquid

$d$  = depth of ring below undisturbed free surface

$a$  = radius of right circular cylinder

$w$  = ring width

References: (32, 35, and 41).

Ring baffles are most effective against lateral slosh which has large vertical components of fluid velocity near the tank wall.

Investigations by Silveira, Stephens and Leonard (35) indicate that:

1. For a given baffle depth:
  - (a) The damping factor increases as the baffle width increases (fig. 5-10).
  - (b) The frequency increases as the baffle width decreases (fig. 5-15).

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

Ring Baffles (continued)

2. Based on the total surface of a baffle, the highest mean damping factor appears to be afforded by ring baffles as compared to the baffle types depicted in fig. (5-77).

Investigations by Garza and Abramson (32) indicate that:

1. For a given baffle depth:
  - (a) With the hole size constant:
    - (1) The frequency increases for increasing percent perforation (fig. 5-2).
    - (2) The damping factor increases for decreasing percent perforation (fig. 5-8).
  - (b) With the percent perforation constant:
    - (1) The frequency increases for increasing hole size (fig. 5-3).
    - (2) The damping factor increases for decreasing hole size (fig. 5-7).
2. Damping produced by perforated baffles is consistently lower than that produced by a solid baffle (fig. 5-7 and 5-8).

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3 Ring Baffles. Because of the complexity of the boundary conditions, a theoretical approach for predicting the damping of a device is extremely limited. In particular, a fixed ring baffle is the only damping device for which an analytical approach exists. For this reason, it is necessary to resort to an experimental approach consisting of ground oscillation testing of a full scale or model tank containing the damping device to be evaluated. If the size of the model is small as compared to the prototype, the use of similitude theory is advisable in order to obtain reasonable similarity of fluid motion. Since it is generally not possible to model equal values of all dimensionless parameters affecting the fluid motion, an appropriate nondimensional modeling parameter must be chosen. Viscous forces and surface tension are small and can therefore be neglected whereas, forces resulting from both the velocity of the fluid and the inertial effects of acceleration cannot. Acceleration is of especial importance because of the three following effects on slosh phenomena:

1. During ground tests, the eigen frequencies remain constant, whereas, during flight conditions they vary as the square root of the longitudinal acceleration.
2. The wave amplitudes are approximately inversely proportional to the longitudinal acceleration.
3. The damping factor is proportional to the square root of the liquid amplitude.
4. The force and moment resultants are proportional to  $g$ .

Therefore, Froude's number, being the ratio of fluid velocity forces to inertia forces, is selected for the modeling parameter.

## V. Propellant Slop Suppression

## 5.1 Anti-Slop Devices (continued)

5.1.3.1 Ring Baffles: Experimental. Each of Articles 5.1.3.1.1 through 5.1.3.1.3 presents an analysis concerned with a particular reference from which experimental data in the form of graphs is presented. The analyses are composed of two parts; the first being concerned with experimental procedure and the second with interpretation of results. This information is meant to aid the designer in both evaluating data of a particular source and comparing data of different sources. Also, this information should be of use in similitude applications since most of the data is obtained from model studies.

5.1.3.1.1 Investigations by Garza and Abramson (32) are outlined below.

Fixed Ring Baffles

1. Baffle thickness: 0.018" to 0.030".
2. Ratio of baffle width to tank radius,  $\frac{W}{R} = 0.157$ .
3. Perforation:
  - (a) The hole size was 0.079" while the percent perforation was varied.
  - (b) The percent perforation was 30% while the hole size was varied.
4. Attachment to container: The baffles were secured to the tank by four 1/8" x 1/3" steel strips attached at 90° apart around the outer edge of the ring. They were supported through angle iron brackets bolted to the upper flange of the tank. The outer edge of the ring was turned down,

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

giving the ring an L type cross section which stiffened the ring considerably, but had little effect on damping (see Figure 5-1).

Container, Excitation and Instrumentation

The container was a rigid-wall circular cylindrical tank supported by four dynamometers. All tests were conducted for three amplitudes of translational excitation. Because of the rather significant effects of excitation amplitude, measurements were made for various values ranging from  $0.00184 \leq \frac{x_0}{d} \leq 0.00823$ , and then all data was presented in terms of RMS values. Table 5-1 gives the resonant frequencies for each of the various tests conducted.

Damping Factor

Experimental values of the damping factor  $\gamma_s$  were obtained from resonant peaks of experimental force response curves. Theoretical values of  $\gamma_s$  were calculated from Miles' equation (Article 5.1.3.2.1) using liquid surface amplitudes  $\zeta_w$ , measured at the tank walls. Table 5-2 gives the damping values for each of the various tests conducted.

5.1.3.2 Investigations by Silveira, Stephens, and Leonard (35) are outlined below.

Baffles (Figure 5-76)

## 1. Baffle thickness

- (a) 12 inch tank: conic section, 1/16" Plexiglas; all others, 1/8" Plexiglas.

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

(b) 30 inch tank: fixed ring, 1/4" Plexiglas.

## 2. Ratio of baffle width to tank radius

(a) 12 inch tank: see Figure 5-77.

(b) 30 inch tank:  $W/R = 0.076$ .

## 3. Perforation

(a) Hole size, 1/8" diameter.

(b) Percent perforation, 50%.

Container Excitation and Instrumentation

Two containers were used: a 12" diameter tank of 1/8" Plexiglas and a 30" aluminum tank of 0.016" walls and 1/2" base. A paddle was used to excite the liquid in the fundamental mode and then removed when the amplitude was sufficient. Liquid response in the 12" tank was sensed by strain gages mounted on the torsion bars which are located between the base platform and the support ring, whereas the response in the 30" tank was measured by a load cell which replaced one of the three platform supports. The output signals from the strain gauges and the load cell were amplified and then fed into a dampometer to measure the damping and frequency of the liquid motion.

Damping Factor

The rate of decay of the moment resulting from the damped liquid oscillations was measured and the damping factor  $\delta$ , representing the decay of oscillation, was defined as



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

$$\delta = \frac{1}{n} \log_e \frac{M_o}{M_n}$$

where  $n$  is the number of cycles over which the decay was measured,  $M_o$  is the magnitude of the initial moment and  $M_n$  is the magnitude of the moment after  $n$  cycles.

General Comments

1. No graph is given for cruciform baffles since, for all but shallow depths, damping is independent of the fluid depth. Test results for the  $90^\circ$  and  $45^\circ$  positions (see Figure 5-77) are as follows: for  $\frac{W}{R} = 0.169$ ,  $\delta_{90} = 0.072$  and  $\delta_{45} = 0.070$  and for  $\frac{W}{R} = 0.337$ ,  $\delta_{90} = 0.156$  and  $\delta_{45} = 0.142$ .
2. All data points presented represent the average of five or more measured values for the given condition.
3. For high values of baffle depth, the curves in Figures 5-9, 5-10, 5-11, 5-66, 5-67, and 5-69 approach the damping values for which no baffle is present.
4. In Figure 5-77, the mean damping factor for a particular baffle was obtained in the following manner: from the graph relating factor to baffle depth, the area under the curve between  $0.084 \frac{d}{R}$  above and  $0.084 \frac{d}{R}$  below the maximum damping was divided by  $0.168 \frac{d}{R}$ .
5. The curves in Figures 5-12 and 5-14 are given by Miles' equation as presented in reference 33. It is

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

$$\delta = 5.66\pi e^{-4.6\frac{d}{R}} \left[ \frac{R^2 - (R-W)^2}{R} \right]^{\frac{3}{2}} \left( \frac{\zeta_w}{R} \right)^{\frac{1}{2}}$$

If  $W^2$  is neglected, then the damping factor becomes:

$$\delta = 5.66\pi e^{-4.6\frac{d}{R}} \alpha^{\frac{3}{2}} \left( \frac{\zeta_w}{R} \right)^{\frac{1}{2}}$$

which is identical to Miles' equation except that the constant multiplier is  $5.66 \pi$  instead of 2.83.

6. For high values of baffle depth, the curves in Figures 5-15, 5-16, 5-64, 5-65, and 5-68 approach, asymptotically, the first resonant frequency for a circular cylindrical tank containing no baffles.

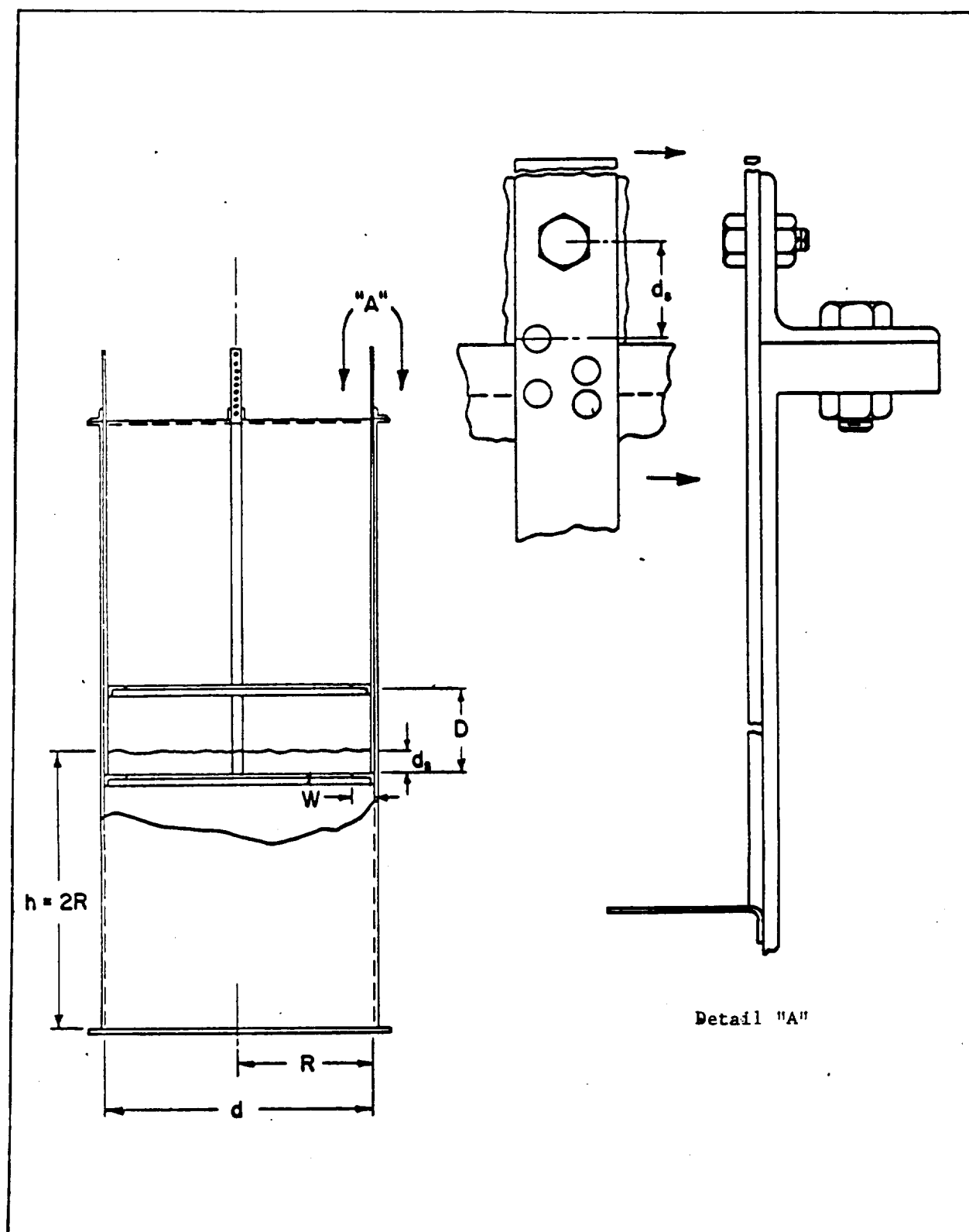


Figure 5.1 Details of baffle support arrangements and tank configuration. (32)

TABLE 5.1

5-14

.. Liquid Resonant Frequency, (32)

$$W/R = 0.157$$

| % Open Area                          | 0                                    | 8%    | 16%   | 23%   | 30%   | 23%   | 30%   | 30%   |
|--------------------------------------|--------------------------------------|-------|-------|-------|-------|-------|-------|-------|
| Hole Diameter<br>(in.)               | 0                                    | 0.079 | 0.079 | 0.079 | 0.079 | 0.040 | 0.040 | 0.020 |
| Ring Depth<br>$d_s/R$                | $\omega^2 d/g$ ( $X_0/d = 0.00184$ ) |       |       |       |       |       |       |       |
| 0                                    | 4.68                                 | 4.40  | 4.25  | 3.57  | 4.1   | 4.11  | 4.14  | 4.33  |
| .025                                 | 4.19                                 | 3.19  | 3.47  | 3.30  | 3.88  | 3.20  | 3.43  | 3.26  |
| .050                                 | 2.92                                 | 3.17  | 3.31  | 3.34  | 3.52  | 3.36  | 3.34  | 3.22  |
| .075                                 | 3.01                                 | 2.90  | 3.27  | 3.32  | 3.47  | 3.26  | 3.34  | 3.32  |
| .100                                 | 3.06                                 | 2.96  | 3.31  | 3.35  | 3.51  | 3.32  | 3.41  | 3.34  |
| .125                                 | 2.97                                 | 3.19  | 3.40  | 3.36  | 3.49  | 3.31  | 3.43  | 3.35  |
| .175                                 | 3.18                                 | 3.24  | 3.35  | 3.32  | 3.54  | 3.38  | 3.47  | 3.41  |
| .250                                 | 3.40                                 | 3.36  | 3.40  | 3.45  | 3.56  | 3.52  | 3.54  | 3.42  |
| .375                                 | 3.48                                 | 3.49  | 3.54  | 3.58  | 3.57  | 3.52  | 3.56  | 3.54  |
| .450                                 | 3.47                                 | 3.54  | 3.62  | 3.58  | 3.58  | 3.59  | 3.62  | 3.54  |
| $\omega^2 d/g$ ( $X_0/d = 0.00417$ ) |                                      |       |       |       |       |       |       |       |
| 0                                    | 4.43                                 | 3.84  | 3.76  | 3.73  | 3.65  | 3.81  | 4.16  | 3.87  |
| .025                                 | 4.27                                 | 3.64  | 3.79  | 3.62  | 3.62  | 3.60  | 3.54  | 3.90  |
| .050                                 | 3.71                                 | 3.29  | 3.40  | 3.54  | 3.50  | 3.44  | 3.66  | 3.55  |
| .075                                 | 2.99                                 | 3.14  | 3.28  | 3.32  | 3.46  | 3.26  | 3.45  | 3.31  |
| .100                                 | 2.94                                 | 3.15  | 3.28  | 3.35  | 3.45  | 3.28  | 3.38  | 3.30  |
| .125                                 | 3.09                                 | 3.20  | 3.22  | 3.35  | 3.45  | 3.26  | 3.40  | 3.31  |
| .175                                 | 3.09                                 | 3.24  | 3.26  | 3.38  | 3.51  | 3.26  | 3.45  | 3.34  |
| .250                                 | 3.18                                 | 3.34  | 3.38  | 3.43  | 3.49  | 3.39  | 3.45  | 3.40  |
| .375                                 | 3.33                                 | 3.46  | 3.40  | 3.43  | 3.42  | 3.52  | 3.45  | 3.44  |
| $\omega^2 d/g$ ( $X_0/d = 0.00833$ ) |                                      |       |       |       |       |       |       |       |
| 0                                    | 3.96                                 | 3.84  | 3.73  | 3.54  | 3.52  | 3.58  | 3.28  | 3.49  |
| .025                                 | 3.77                                 | 3.77  | 3.75  | 3.49  | 3.50  | 3.54  | 3.41  | 3.54  |
| .050                                 | 3.58                                 | 3.58  | 3.51  | 3.49  | 3.47  | 3.47  | 3.40  | 3.49  |
| .075                                 | 3.18                                 | 3.24  | 3.28  | 3.40  | 3.40  | 3.40  | 3.41  | 3.40  |
| .100                                 | 2.97                                 | 3.22  | 3.28  | 3.31  | 3.38  | 3.24  | 3.34  | 3.36  |
| .125                                 | 3.00                                 | 3.06  | 3.32  | 3.24  | 3.31  | 3.28  | 3.30  | 3.28  |
| .175                                 | 3.07                                 | 3.13  | 3.28  | 3.31  | 3.31  | 3.28  | 3.30  | 3.32  |
| .250                                 | 3.16                                 | 3.28  | 3.28  | 3.31  | 3.28  | 3.22  | 3.32  | 3.32  |
| .375                                 | 3.30                                 | 3.32  | 3.31  |       | 3.24  | 3.30  | 3.36  | 3.32  |

TABLE 5.2

5-15

Ring Damping. (32)  
 $W/R = 0.157$

| % Open Area                      | 0                                | 8%    | 16%   | 23%   | 30%   | 23%   | 30%   | 30%   |
|----------------------------------|----------------------------------|-------|-------|-------|-------|-------|-------|-------|
| Hole Diameter<br>(in.)           | 0                                | 0.079 | 0.079 | 0.079 | 0.079 | 0.040 | 0.040 | 0.020 |
| Ring Depth<br>$d_s/R$            | Damping Ratio, $X_0/d = 0.00184$ |       |       |       |       |       |       |       |
| 0                                | .060                             | .087  | .084  | .108  | .087  | .091  | .110  | .103  |
| .025                             | .210                             | .132  | .084  | .106  | .087  | .114  | .112  | .116  |
| .050                             | .137                             | .147  | .071  | .073  | .055  | .086  | .086  | .103  |
| .075                             | .090                             | .1145 | .070  | .065  | .050  | .072  | .082  | .084  |
| .100                             | .070                             | .0885 | .069  | .059  | .045  | .046  | .050  | .057  |
| .125                             | .073                             | .067  | .056  | .049  | .045  | .059  | .045  | .053  |
| .175                             | .060                             | .054  | .056  | .045  | .040  | .054  | .033  | .049  |
| .250                             | .035                             | .040  | .035  | .047  | .031  | .032  | .032  | .042  |
| .375                             | .027                             | .025  | .026  | .021  | .022  | .023  | .022  | .029  |
| .450                             | .025                             | .022  |       |       | .017  | .023  | .015  | .023  |
| Damping Ratio, $X_0/d = 0.00417$ |                                  |       |       |       |       |       |       |       |
| 0                                | .101                             | .127  | .101  | .098  | .082  | .094  | .102  | .086  |
| .025                             | .121                             | .126  | .122  | .108  | .093  | .110  | .114  | .109  |
| .050                             | .173                             | .134  | .119  | .112  | .089  | .109  | .106  | .103  |
| .075                             | .160                             | .112  | .108  | .084  | .077  | .104  | .077  | .089  |
| .100                             | .110                             | .091  | .090  | .075  | .064  | .085  | .068  | .085  |
| .125                             | .077                             | .102  | .077  | .064  | .047  | .077  | .064  | .078  |
| .175                             | .072                             | .070  | .072  | .061  | .050  | .066  | .054  | .070  |
| .250                             | .051                             | .061  | .051  | .049  | .047  | .051  | .049  | .055  |
| .375                             | .039                             | .044  | .038  | .035  | .040  | .039  | .032  | .037  |
| Damping Ratio, $X_0/d = 0.00833$ |                                  |       |       |       |       |       |       |       |
| 0                                | .117                             | .122  | .121  | .111  | .095  | .110  | .091  | .114  |
| .025                             | .146                             | .126  | .124  | .117  | .087  | .122  | .106  | .112  |
| .050                             | .177                             | .134  | .124  | .109  | .098  | .126  | .098  | .118  |
| .075                             | .158                             | .123  | .114  | .099  | .084  | .104  | .086  | .120  |
| .100                             | .142                             | .115  | .102  | .093  | .082  | .093  | .072  | .093  |
| .125                             | .116                             | .106  | .093  | .088  | .078  | .093  | .082  | .088  |
| .175                             | .096                             | .096  | .081  | .078  | .072  | .079  | .074  | .072  |
| .250                             | .082                             | .066  | .067  | .061  | .057  | .067  | .058  | .058  |
| .375                             | .066                             | .048  | .046  |       | .026  | .046  | .038  | .042  |

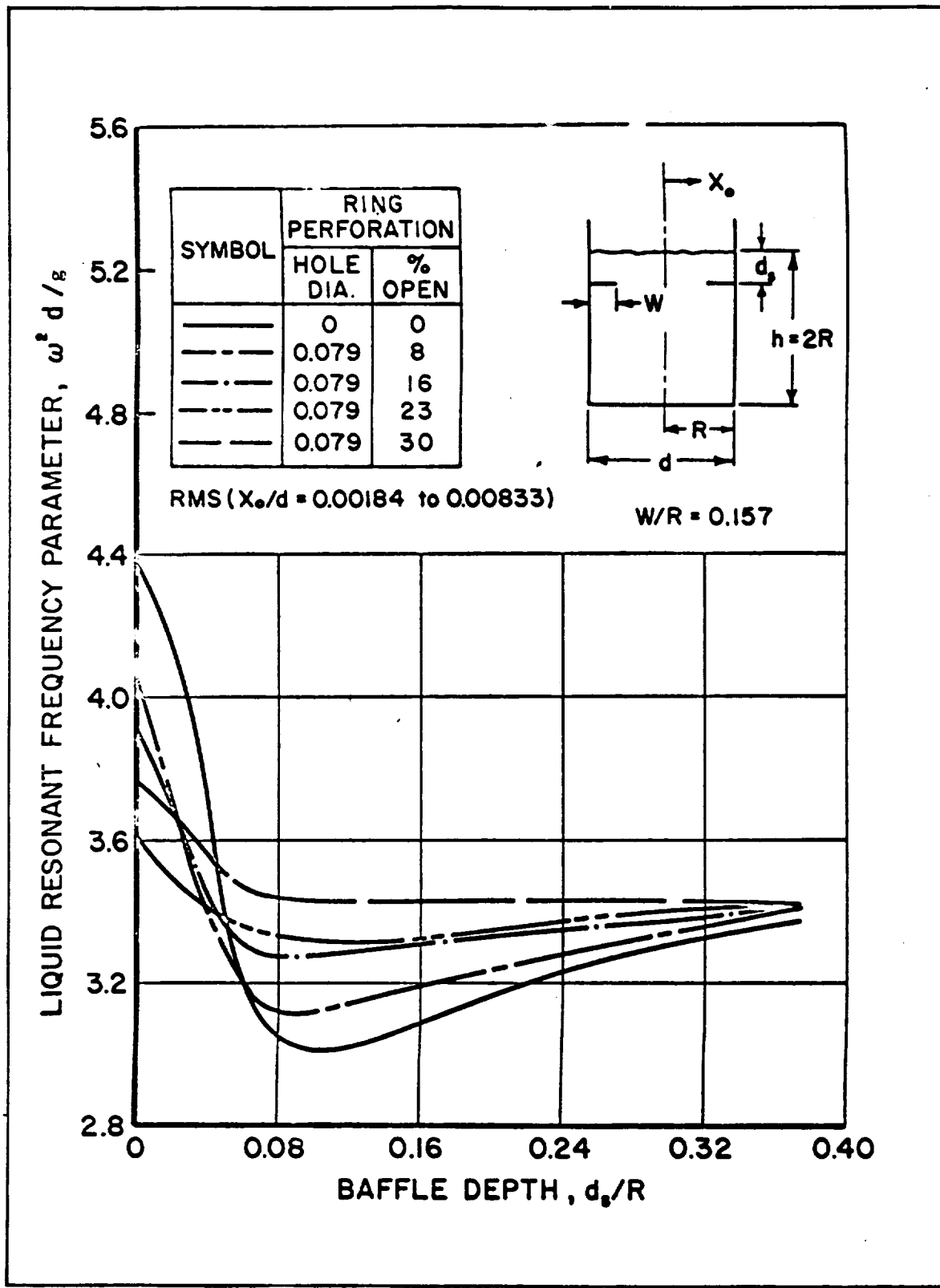


Figure 5.2 Effect of percent perforation on liquid resonant frequencies as a function of baffle depth. (32)

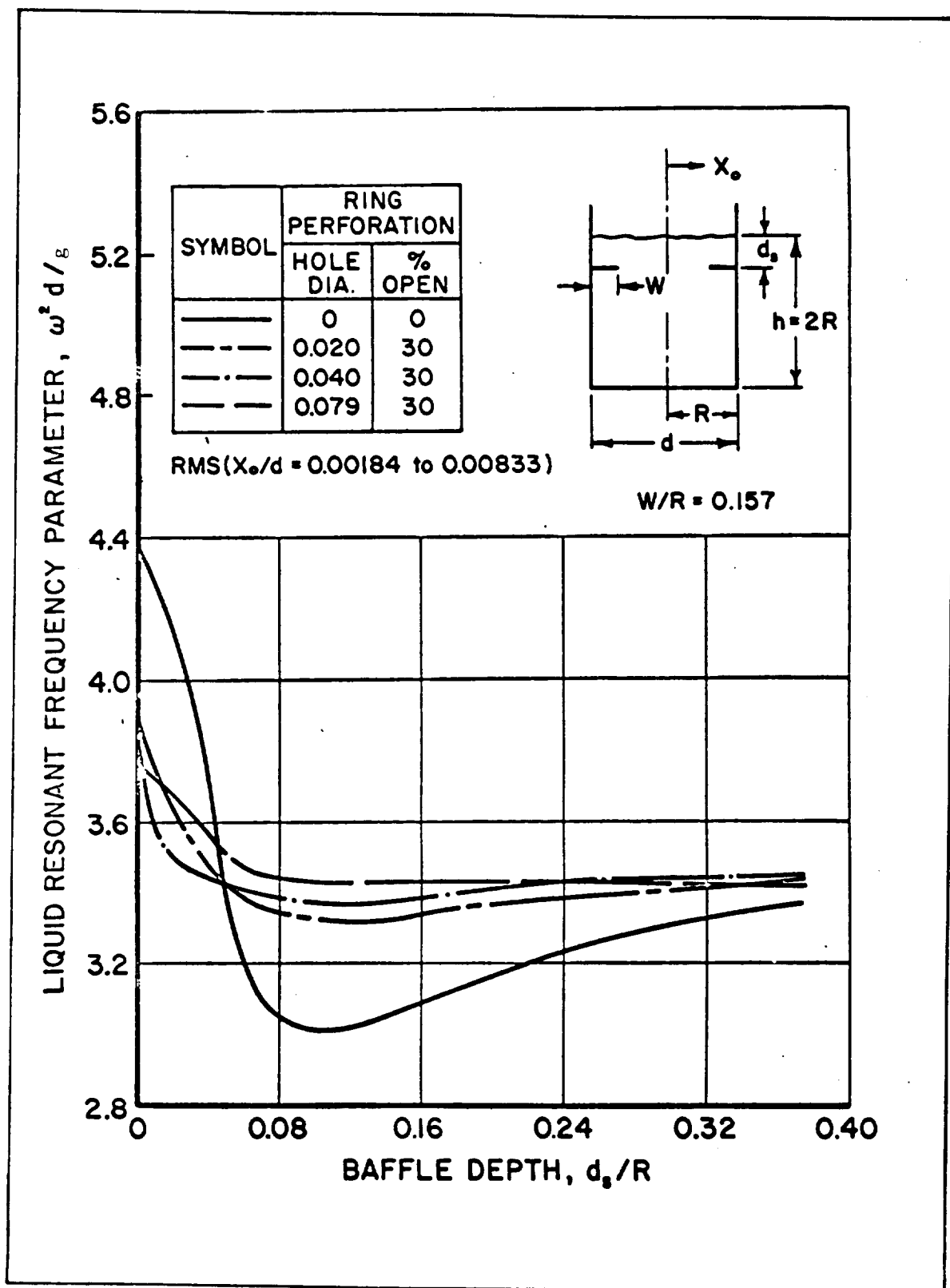


Figure 5.3 Effect of perforation hole size on liquid resonant frequencies as a function of baffle depth. (32)

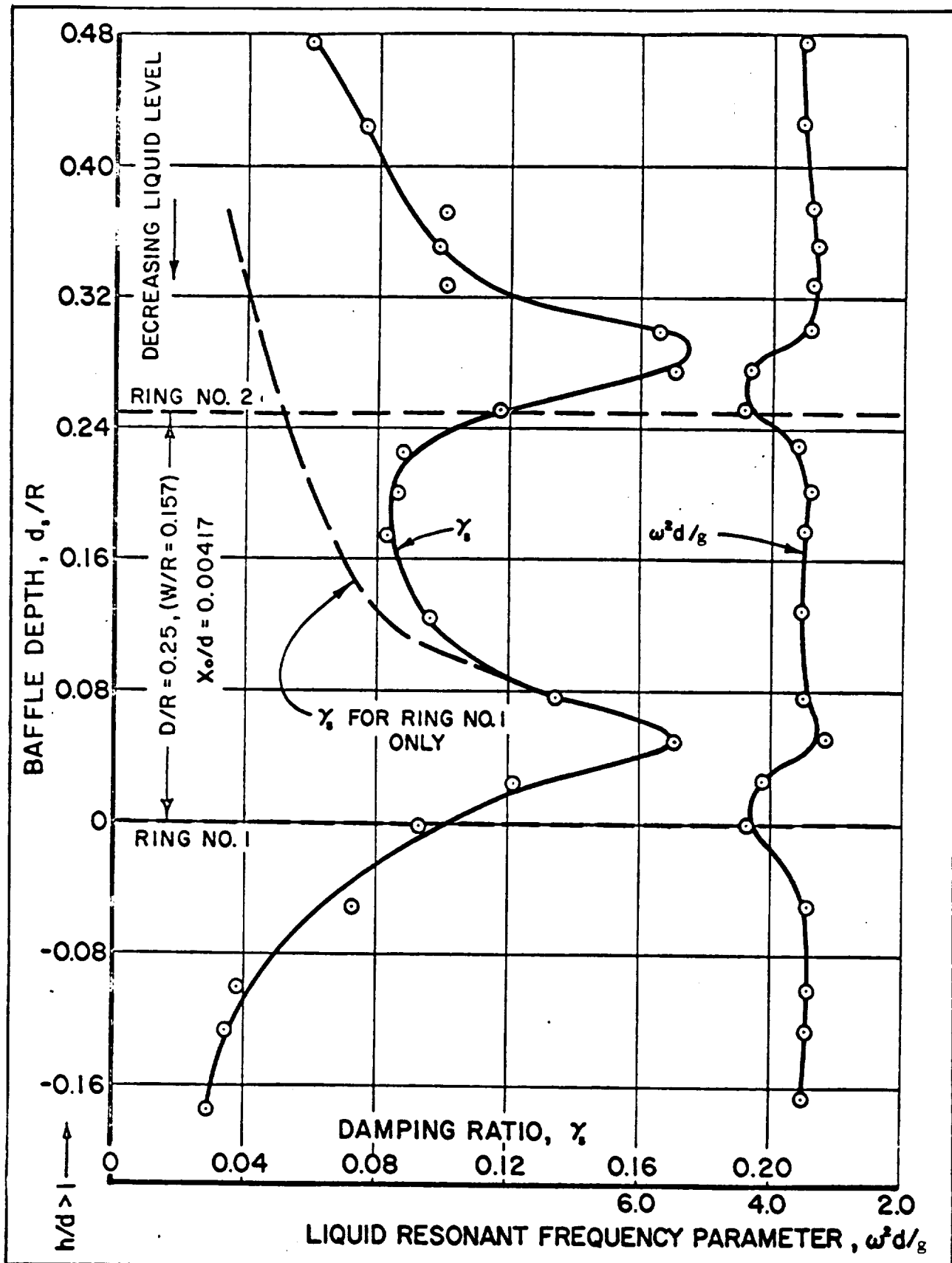


Figure 5.4 Effects of double rings as a function of baffle depth. (32)



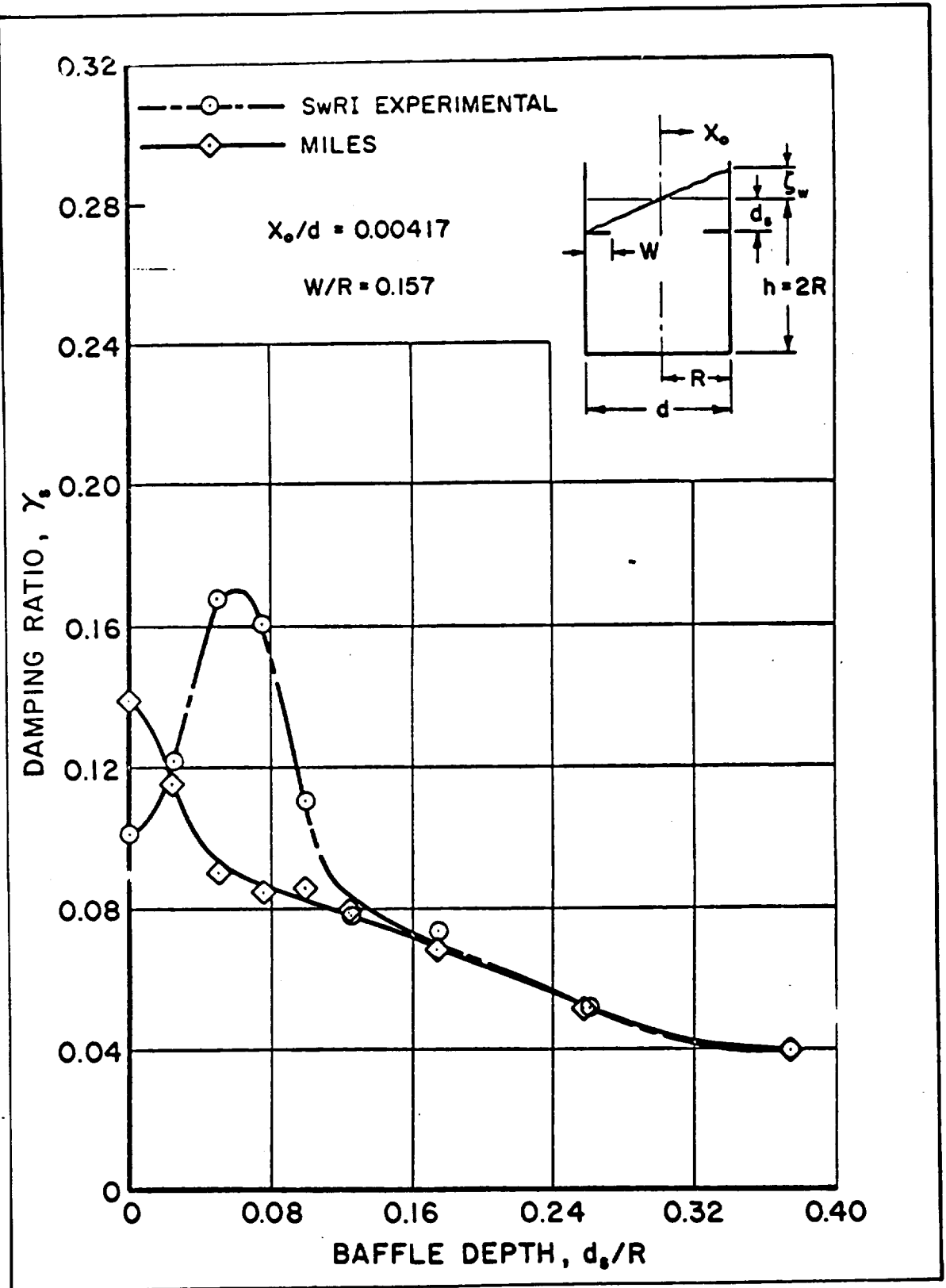


Figure 5.5 Comparison of theory and experiment for damping provided by a flat solid ring baffle as a function of baffle depth. (32)

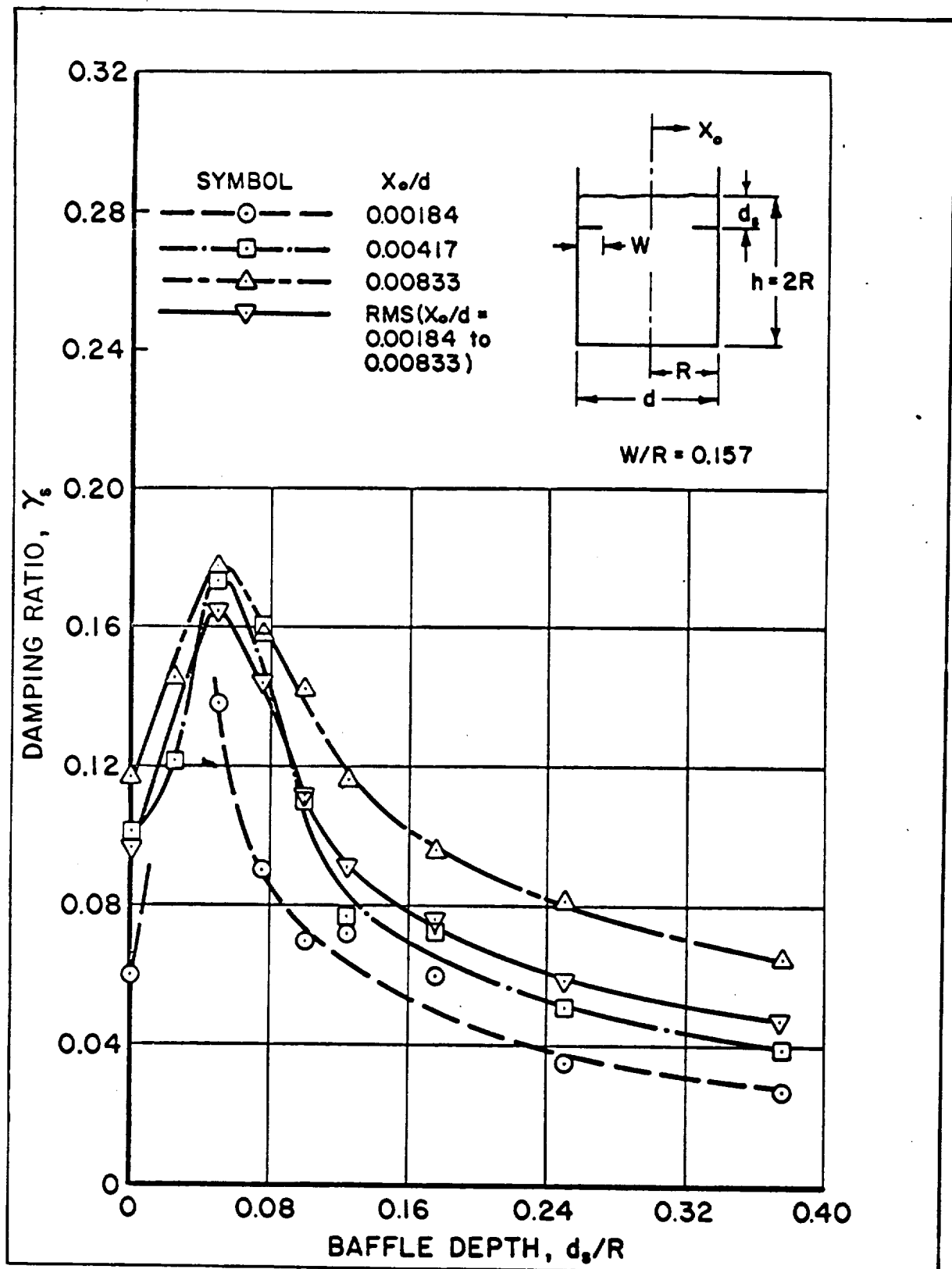


Figure 5.6 Effect of excitation amplitude on damping effectiveness of a solid flat ring baffle as a function of baffle depth. (32)

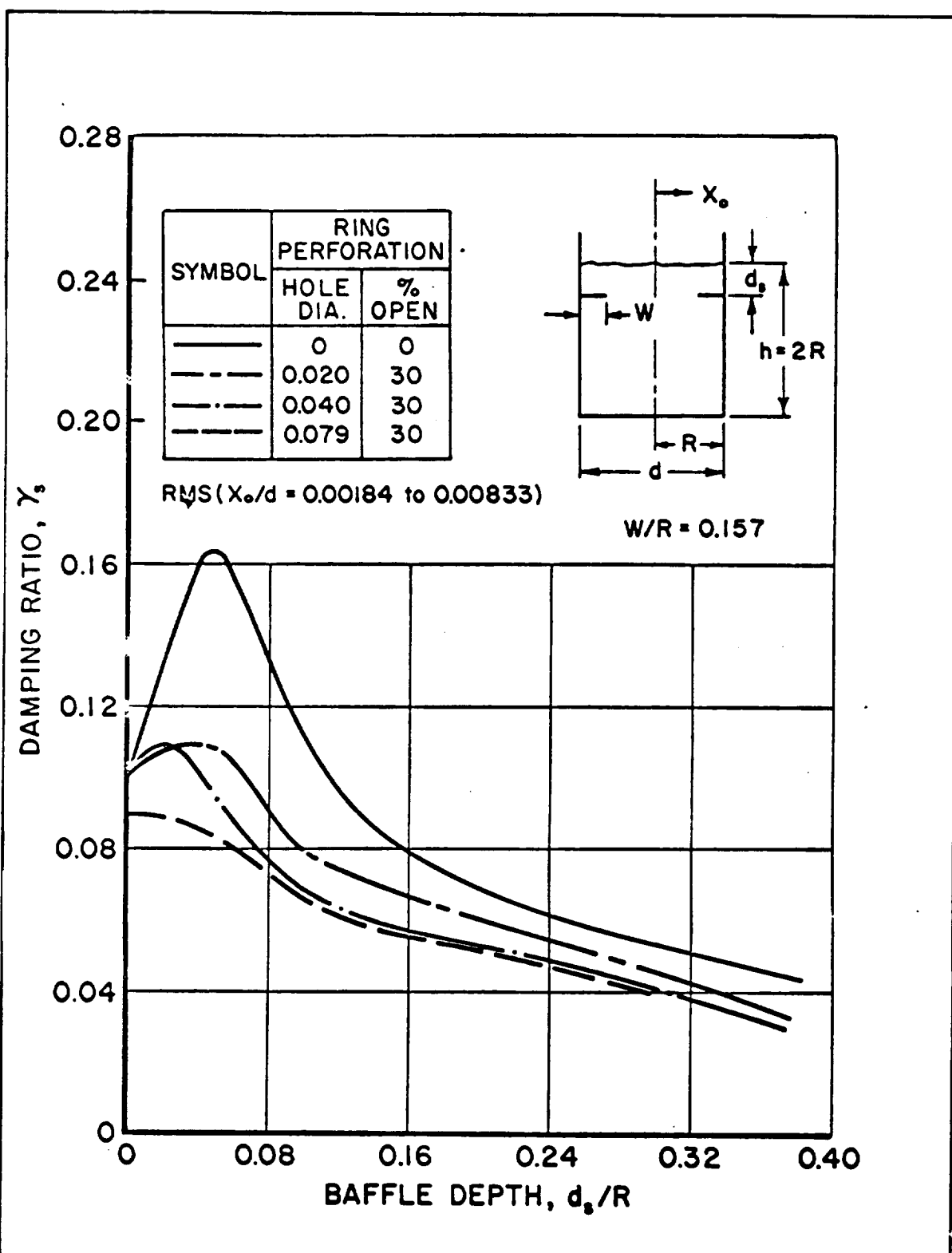


Figure 5.7 Effect of perforation hole size on damping effectiveness as a function of baffle depth. (32)

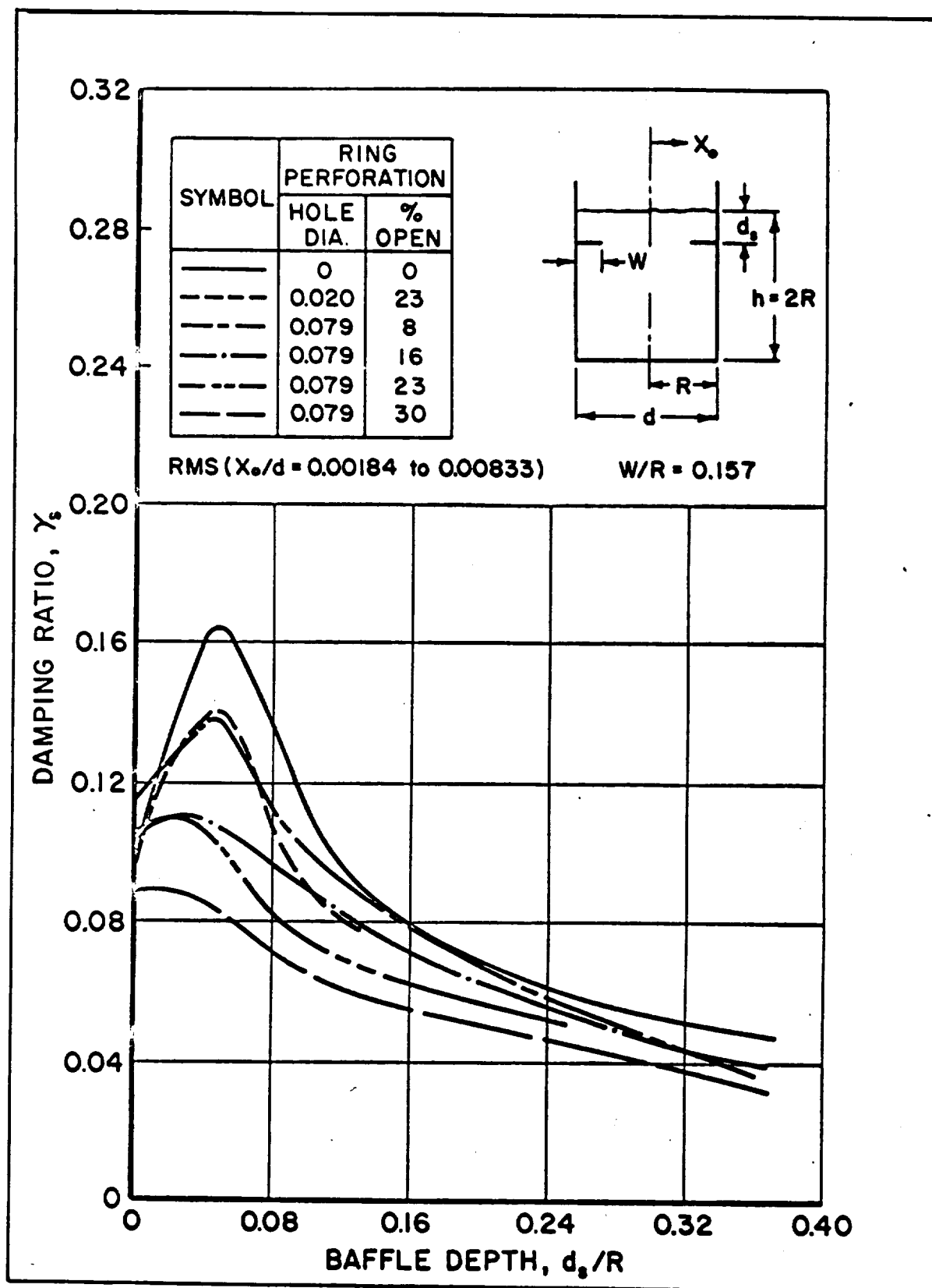
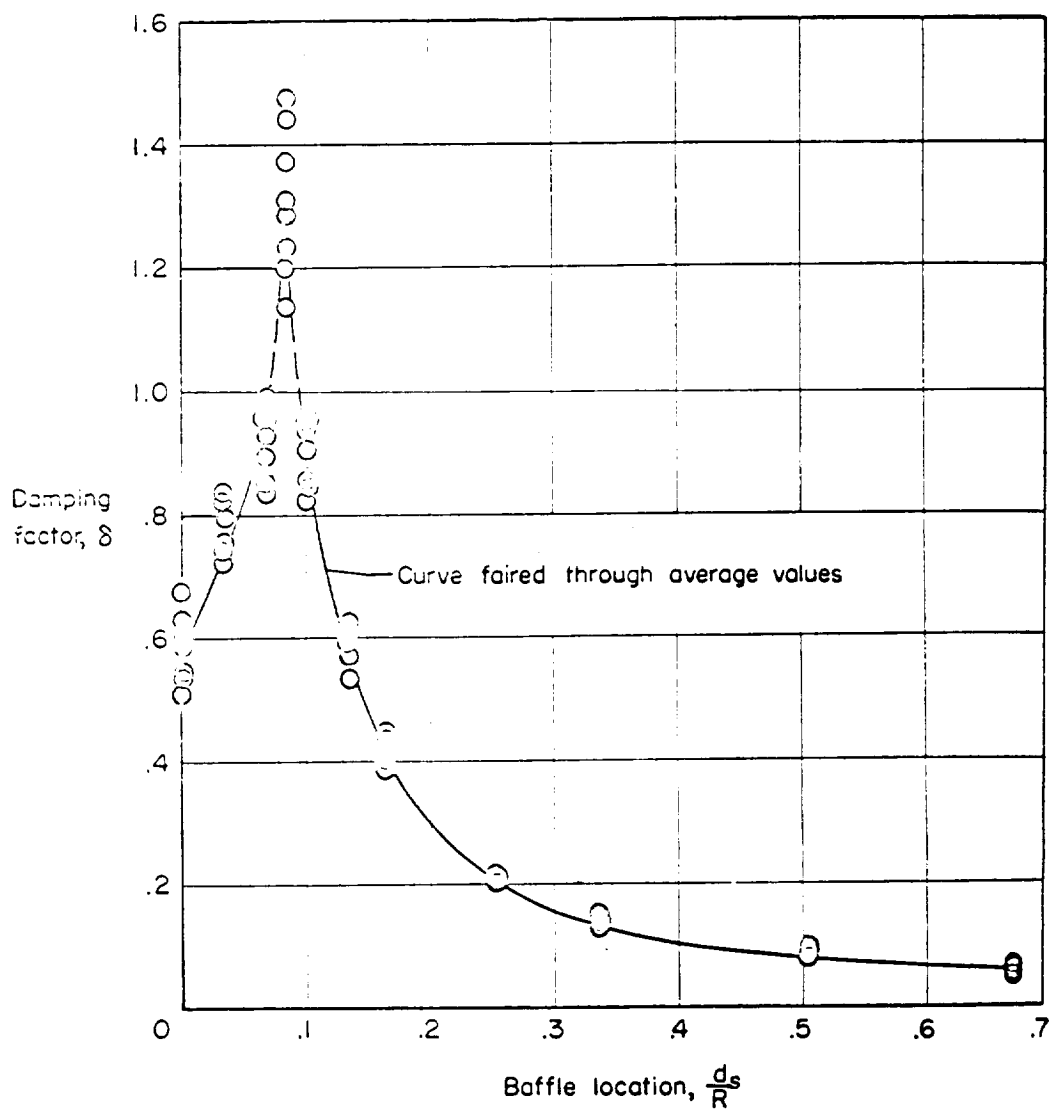


Figure 5.8 Effect of percent perforation on damping effectiveness as a function of baffle depth. (32)



A typical distribution of data for a fixed-ring baffle as a function of baffle depth.  $R = 6$  inches;  $\frac{h}{R} = 2$ . (35)

Figure 5-9

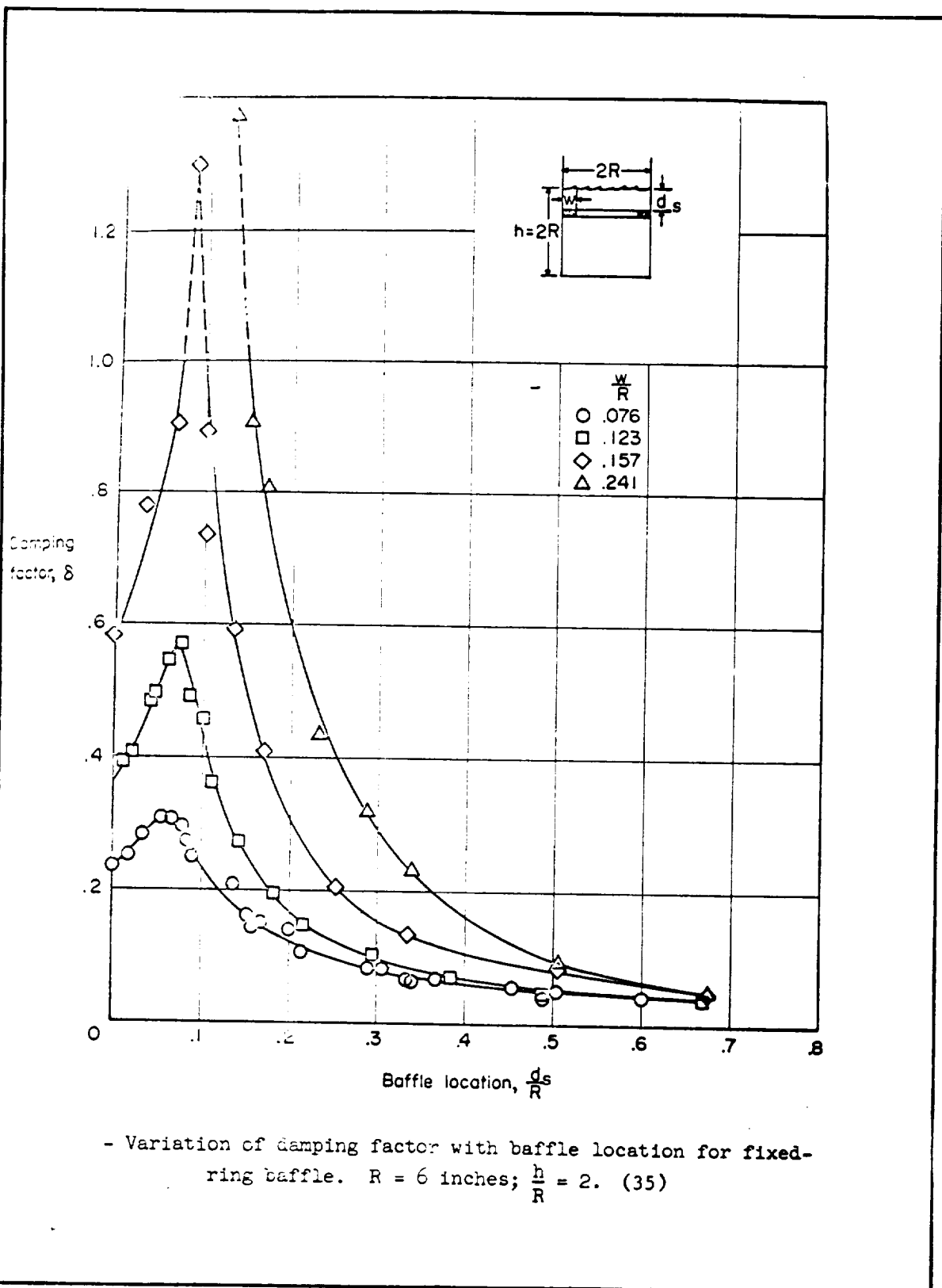
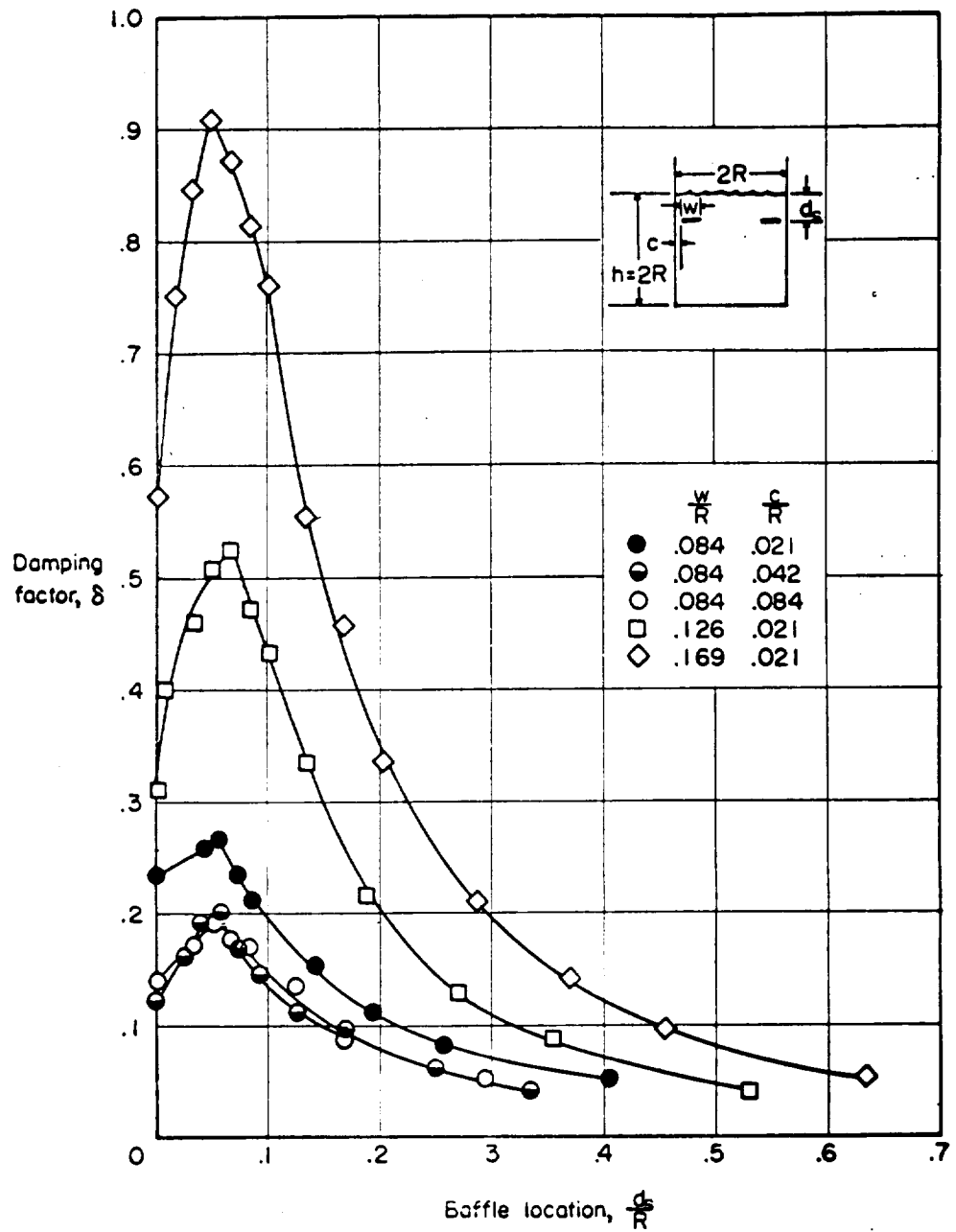
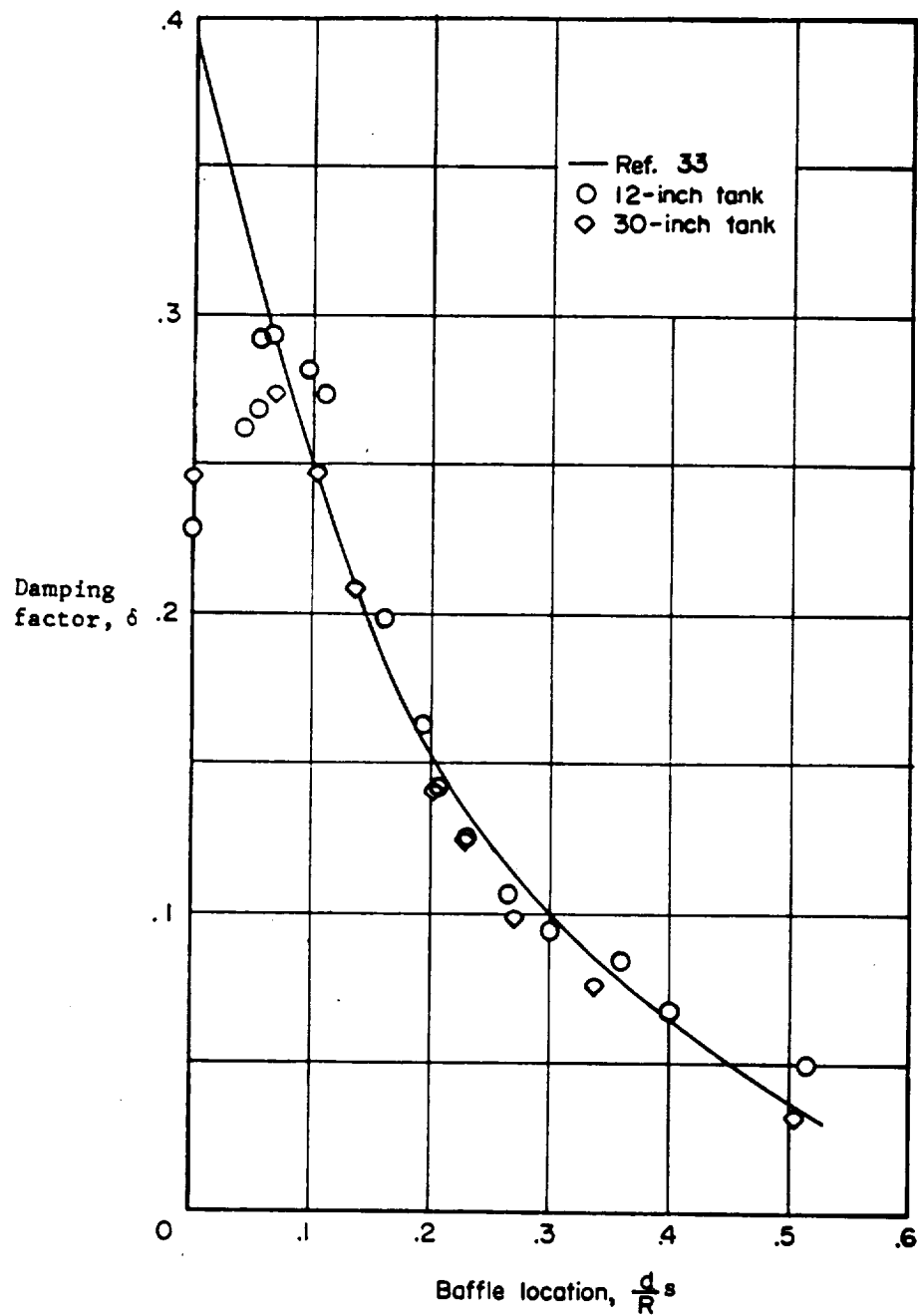


Figure 5-10



Variation of damping factor with baffle location for ring-with-radial-clearance baffle.  $R = 6$  inches;  $\frac{h}{R} = 2. (35)$

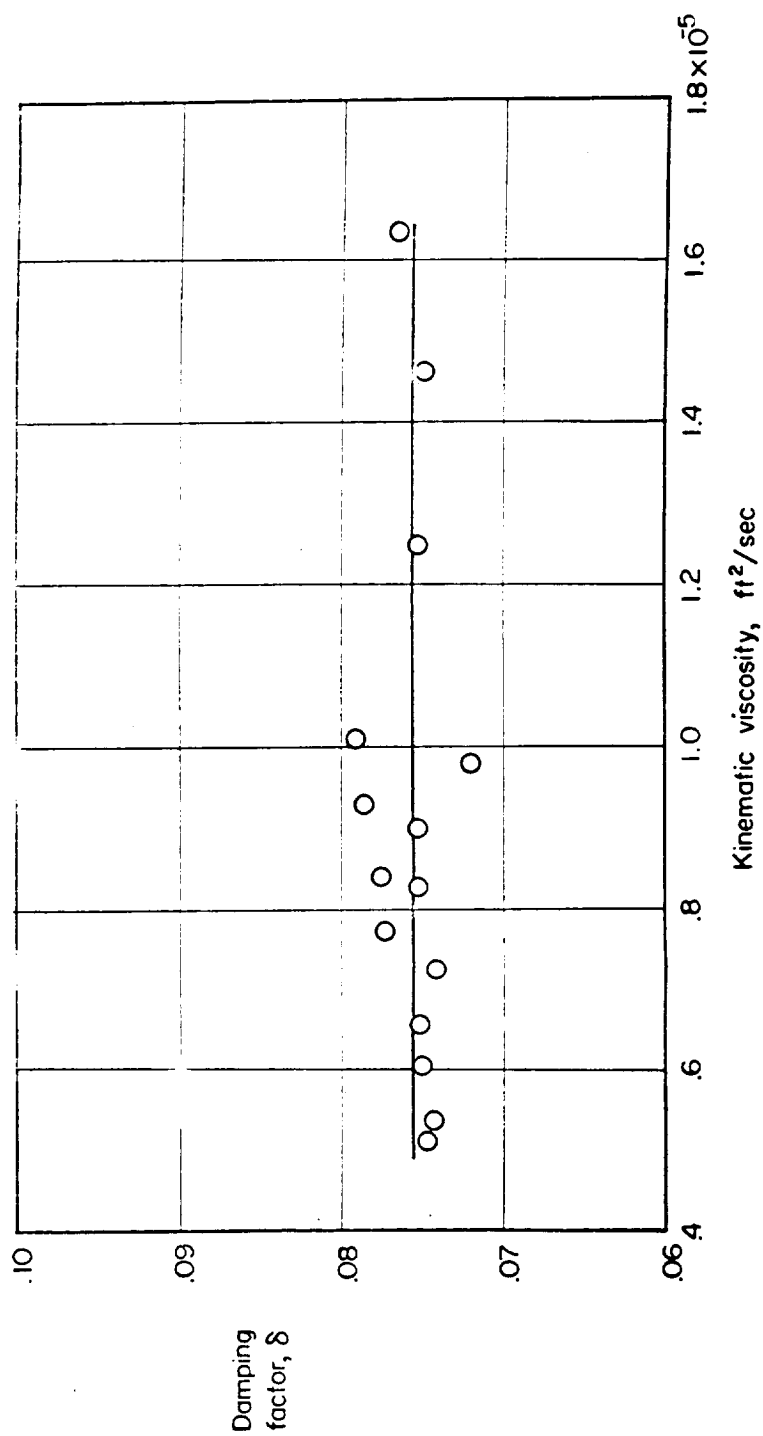
Figure 5-11



Variation of damping factor with baffle location in the 12- and 30-inch-diameter tanks. (35)

Figure 5-12





Variation of damping factor with kinematic viscosity of water for fixed-ring baffle mounted in 30-inch-diameter tank.  $\frac{W}{R} = 0.076$ . (35)



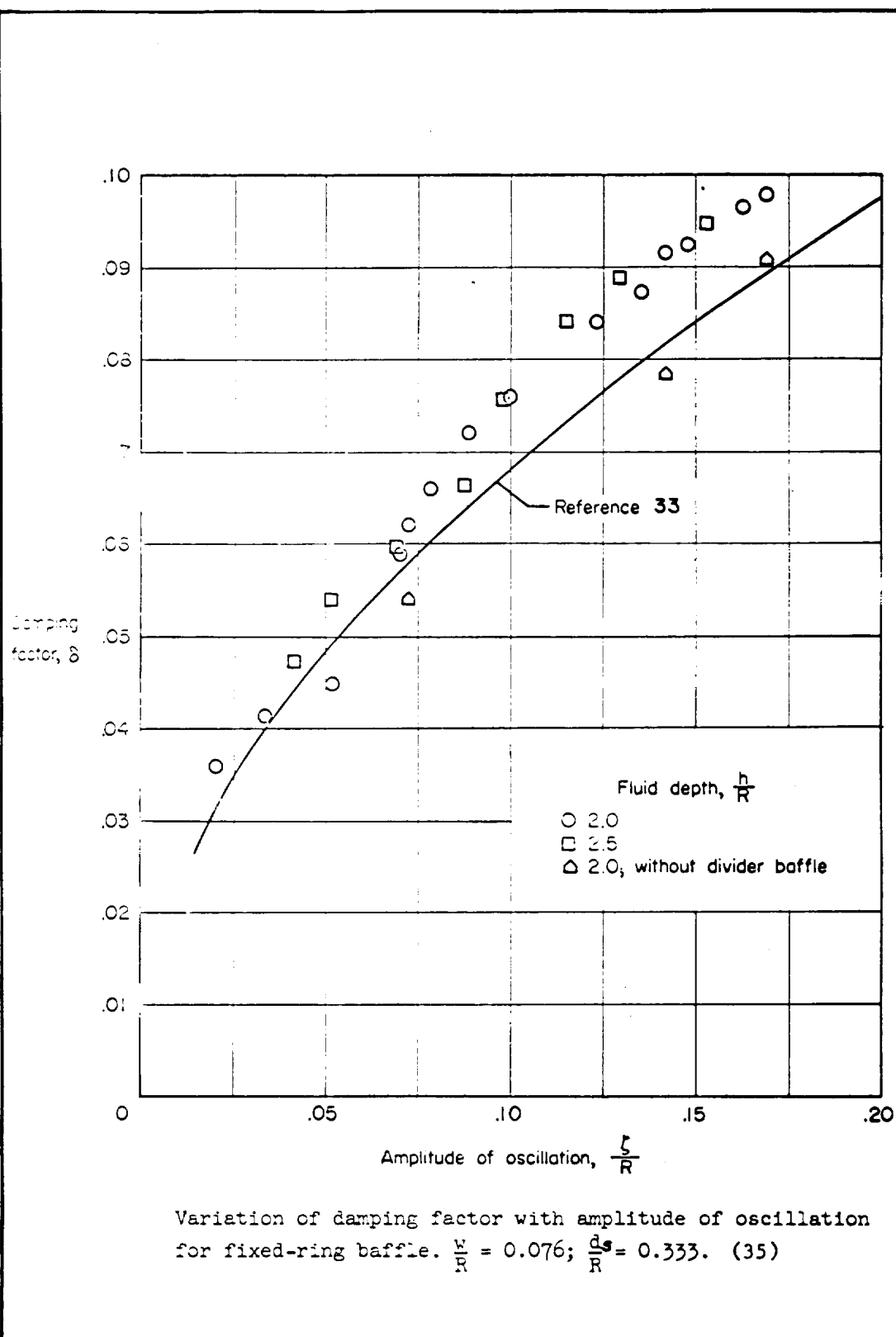
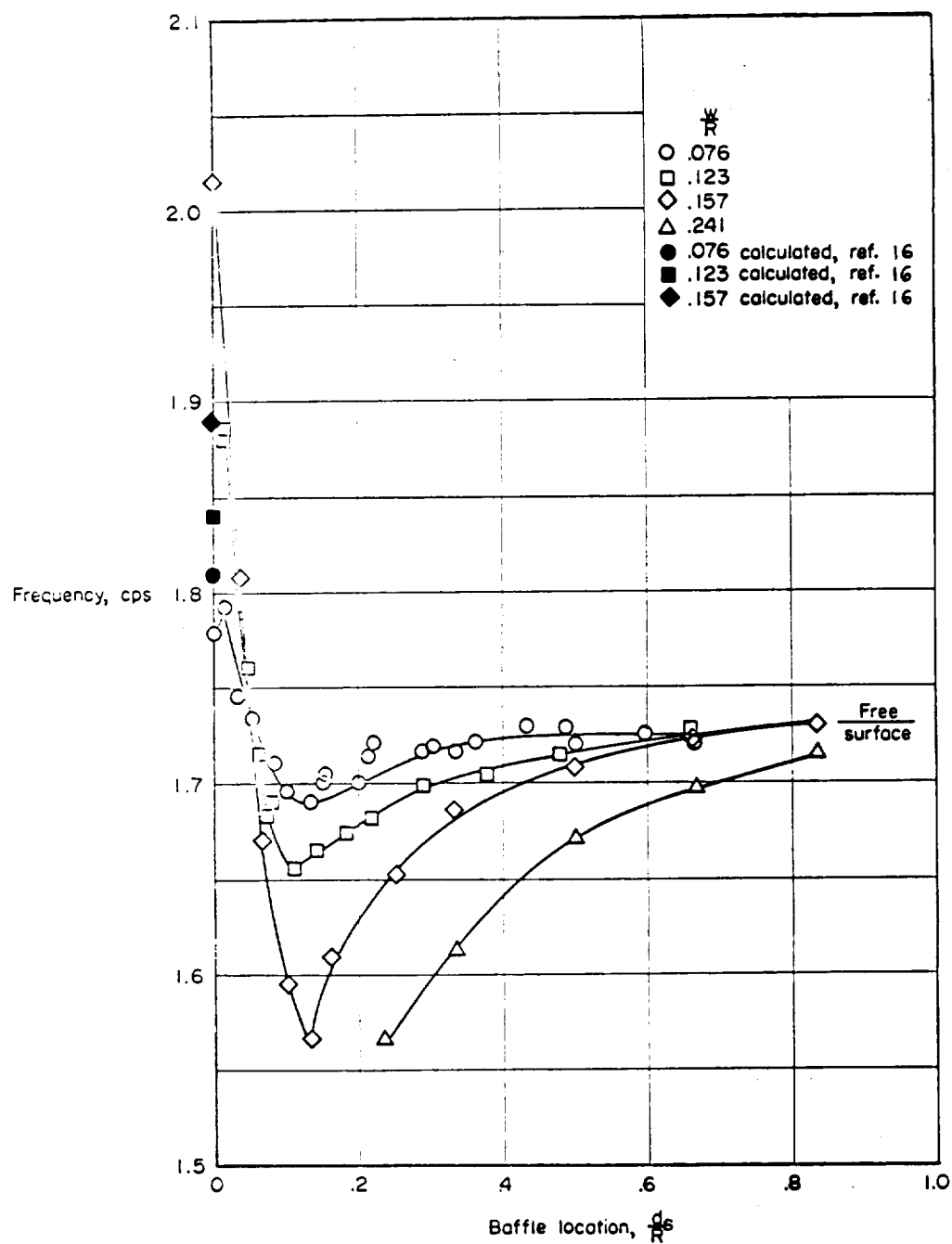


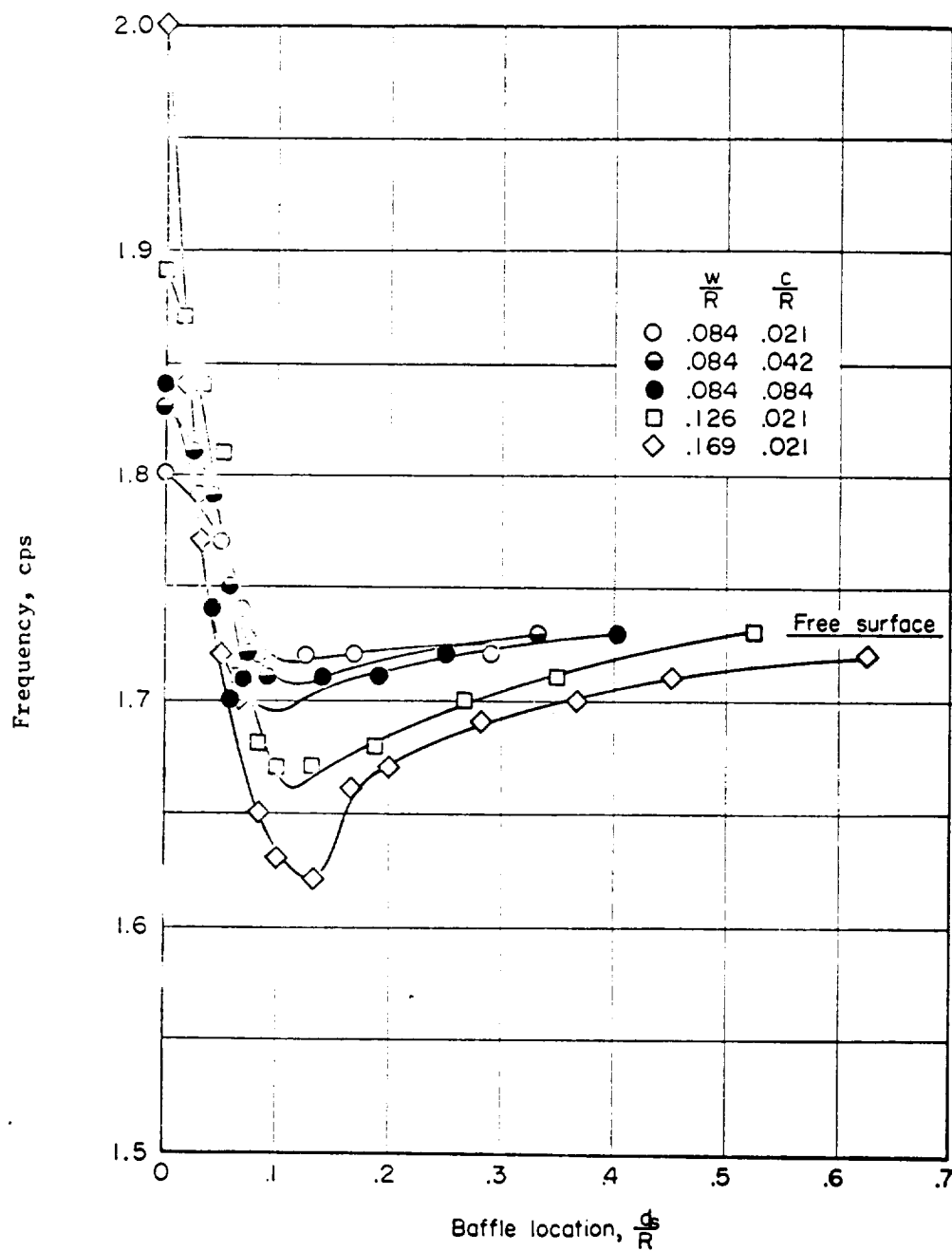
Figure 5-14





Variation of frequency with baffle location for fixed-ring baffle. (35)

Figure 5-15



Variation of frequency with baffle location for ring-with-radial-clearance baffle. (35)

Figure 5-16

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2 Ring Baffles: Analytical. Damping requirements are determined from stability studies of the vehicle in question. The designer must then determine the baffle width and spacing needed to yield a damping factor in the required range.

5.1.3.2.1 Miles' Analysis (28). According to Miles, the damping ratio of an annular ring baffle is

$$\gamma = 2.83 \alpha^{3/2} e^{-\frac{4.6d}{a}} (\zeta_1/a)^{1/2}, \quad \alpha \ll 1 \text{ and } \zeta_1 \ll a \quad (5-1)$$

where

$\alpha \pi a^2$  = ring baffle area

$\alpha$  = ratio of ring baffle area to cross-sectional area of tank

$d$  = depth of ring baffle below undisturbed free liquid surface

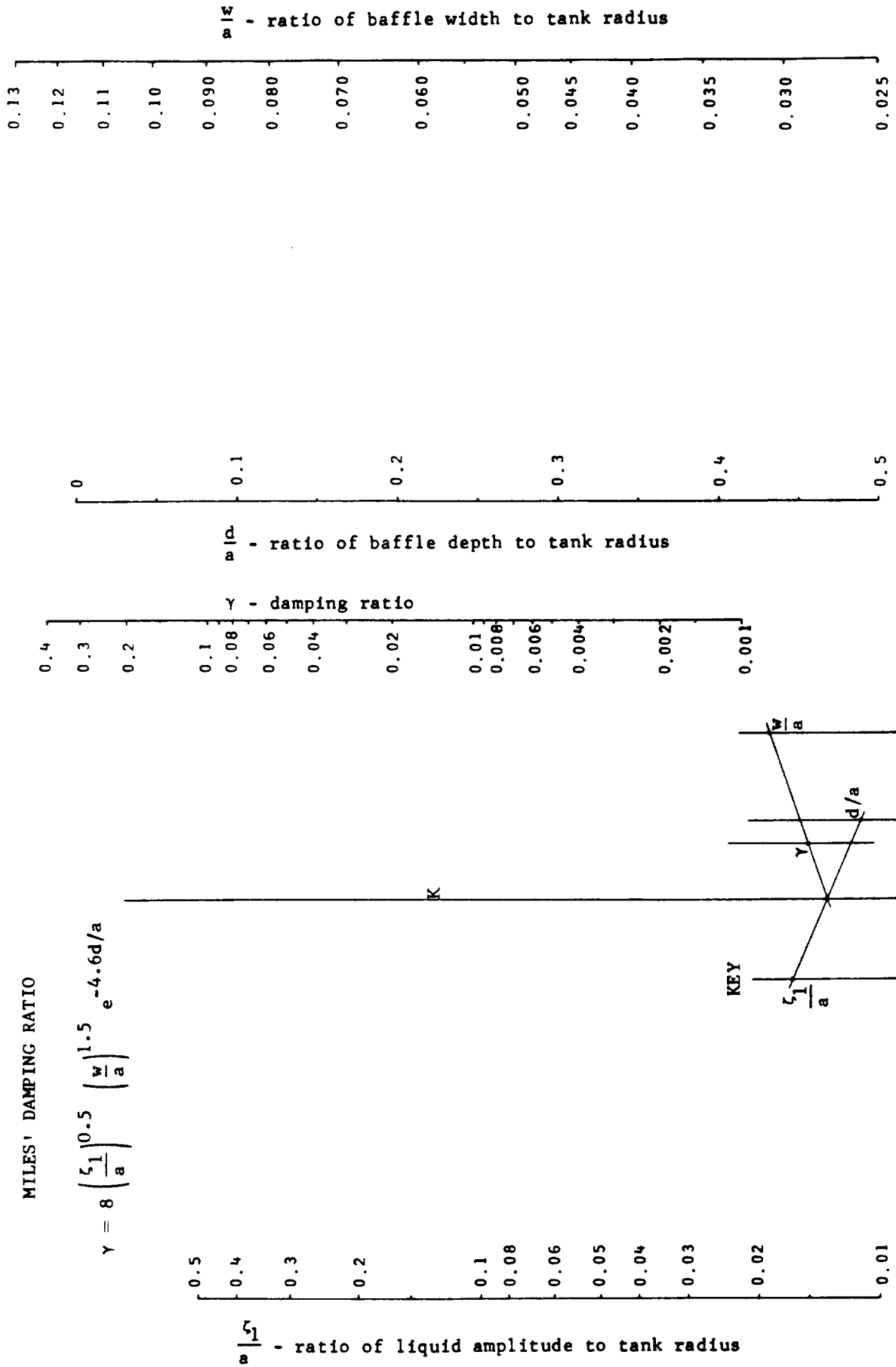
$a$  = radius of circular cylindrical tank

$\zeta_1$  = amplitude of liquid oscillations measured at the tank wall from undisturbed free liquid surface.

Within the restrictions on  $\alpha$  and  $\zeta_1$ , this damping ratio relation was experimentally verified; however, Miles suggested that additional experimental confirmation would be desirable, both in order to establish the limits on  $\alpha$  and  $\zeta_1/a$  and to determine a more accurate value of the constant multiplier. Results are shown in Figures 5-17 to 5-29. A nomograph is shown on page 5-32. A line drawn between values of  $\zeta_1/a$  and  $d/a$  determines a pivot point on the K-axis. A line drawn from this pivot point through a value of  $\gamma$  gives the corresponding value of  $W/a$ .

## MILES' DAMPING RATIO

$$\gamma = 8 \left( \frac{\zeta_1}{a} \right)^{0.5} \left( \frac{w}{a} \right)^{1.5} e^{-4.6d/a}$$





DAMPING FACTOR VERSUS BAFFLE DEPTH FOR  
 LIQUID AMPLITUDE  $\zeta_w = 0.01a$  WITH THE BAFFLE  
 WIDTH AS A PARAMETER (MILES FORMULA)

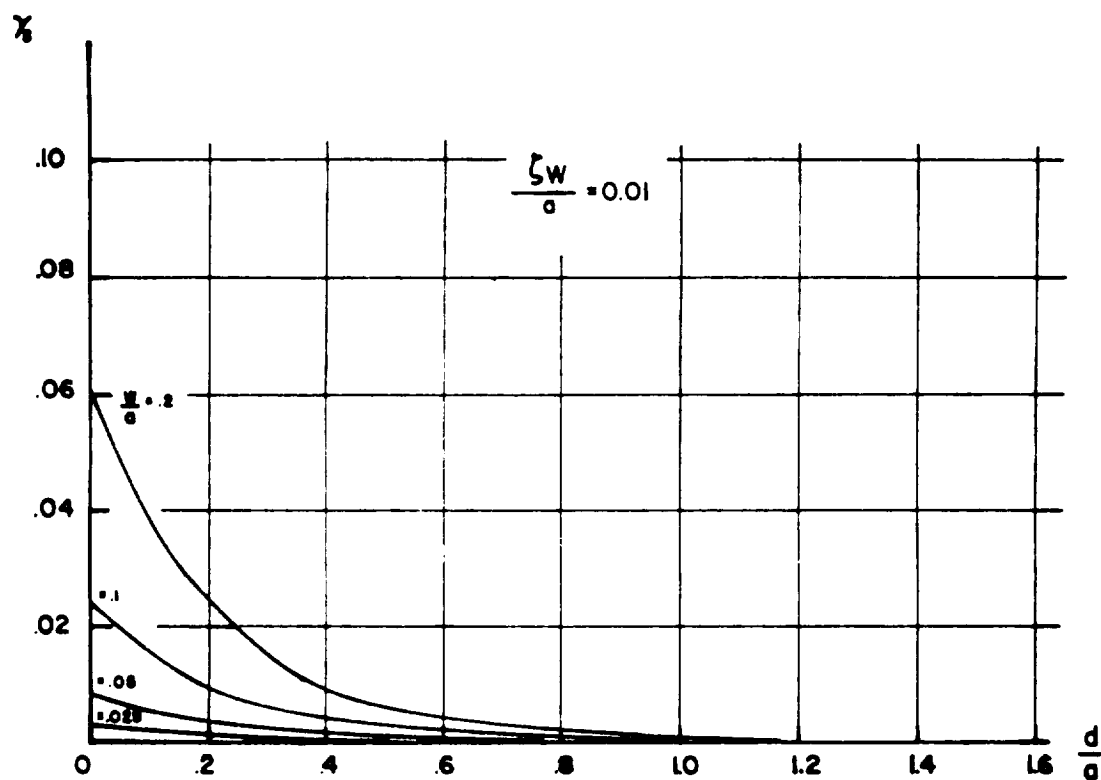


Figure 5-17

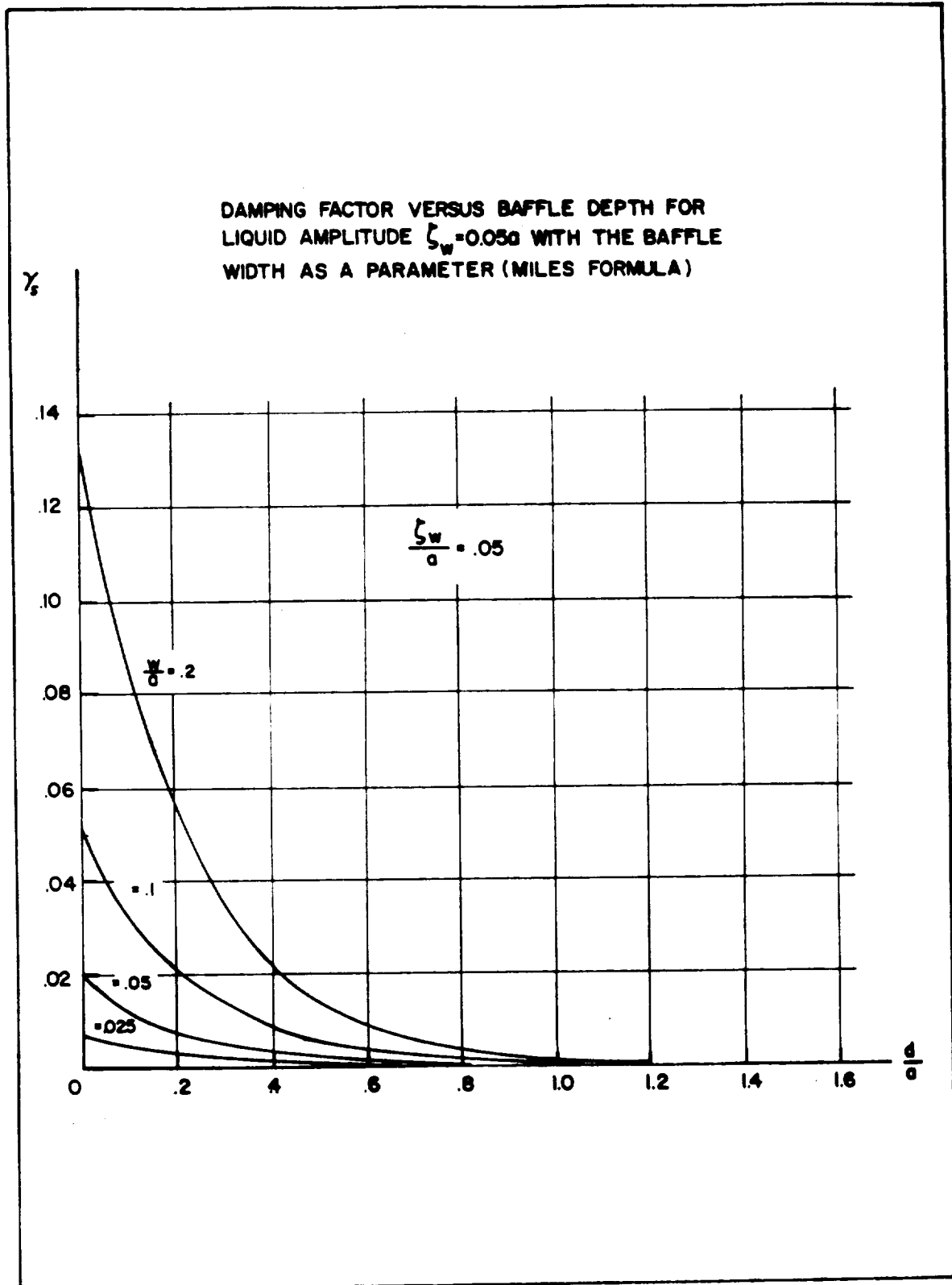


Figure 5-18

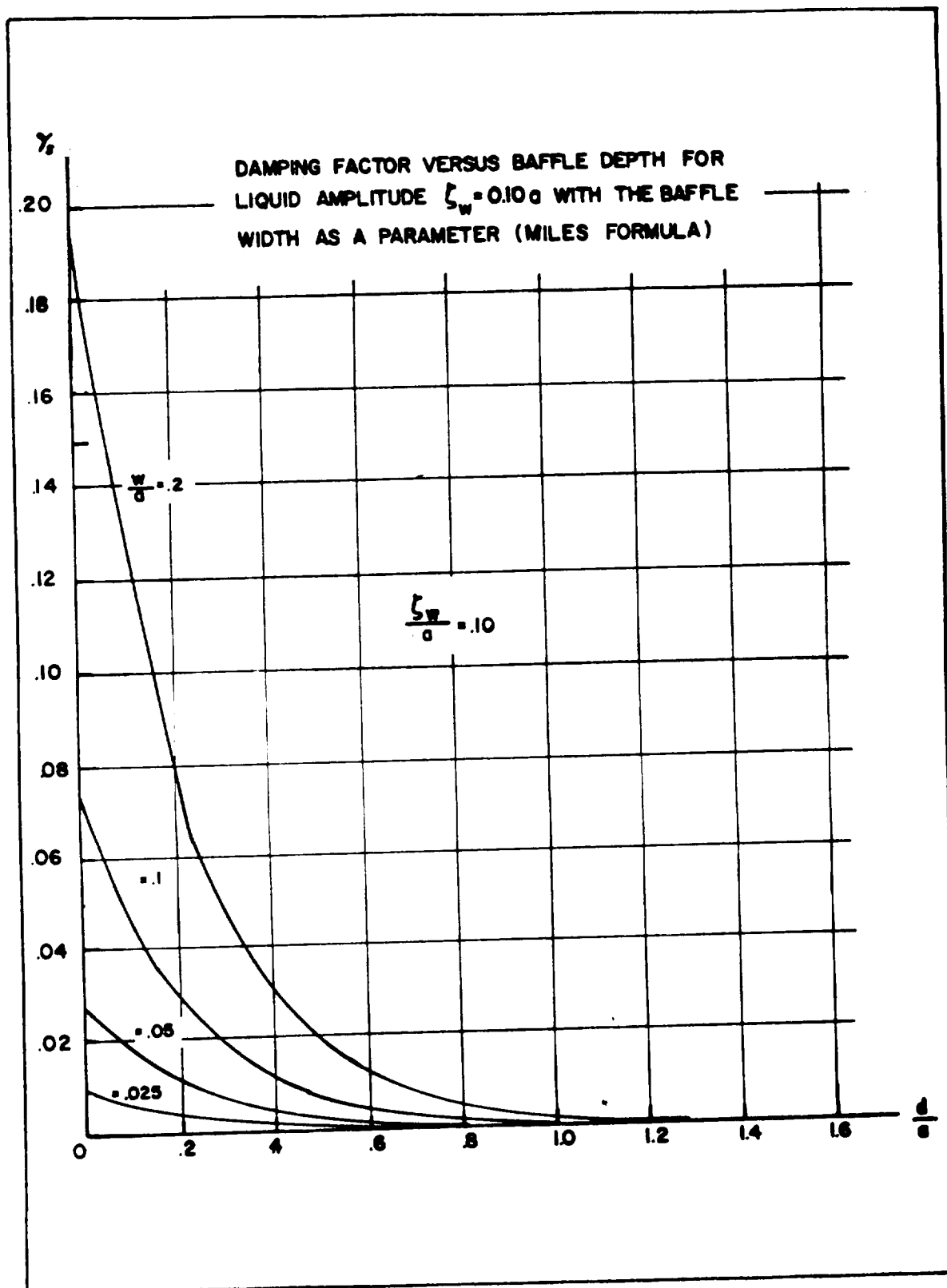


Figure 5-19

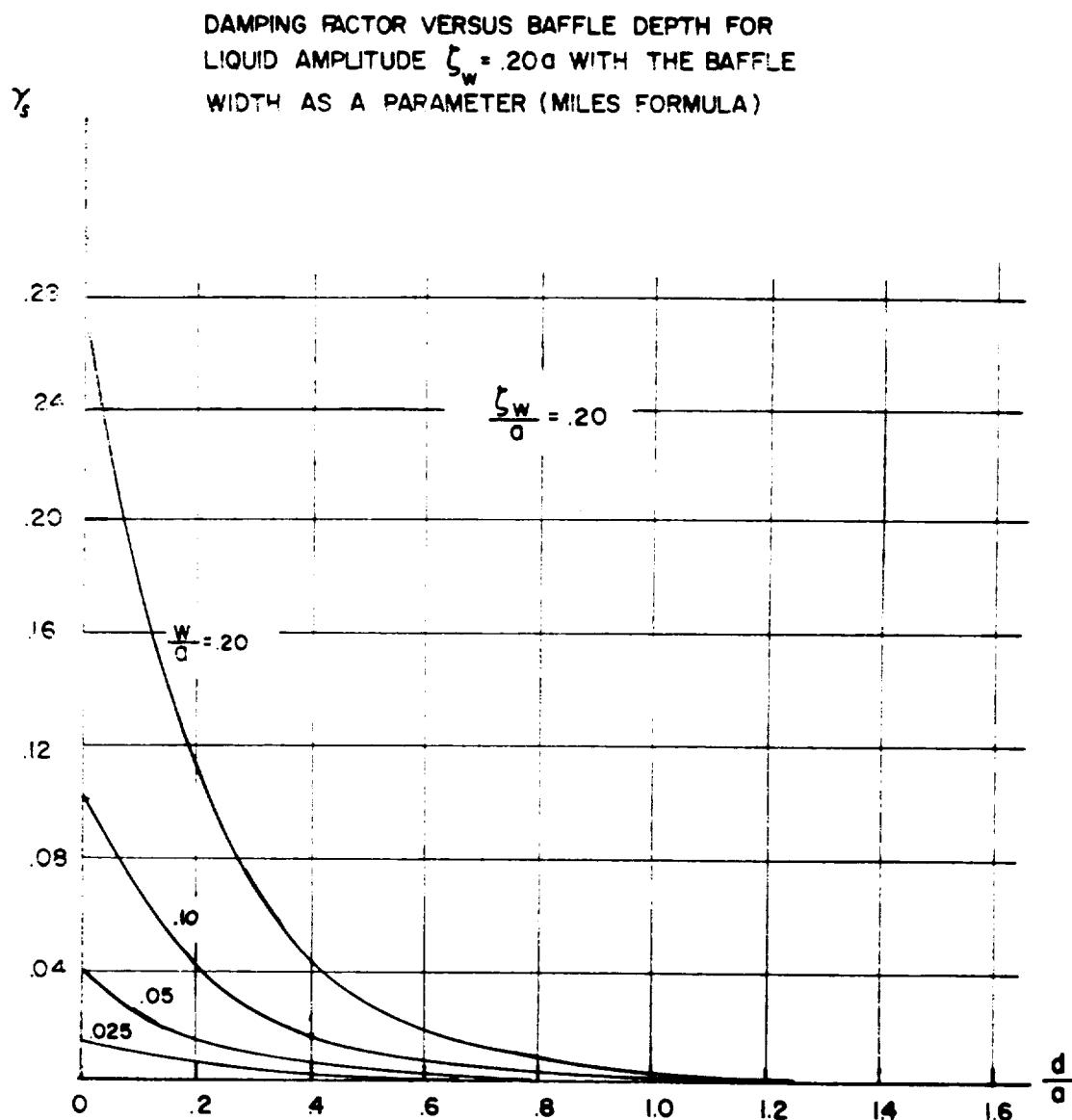


Figure 5-20

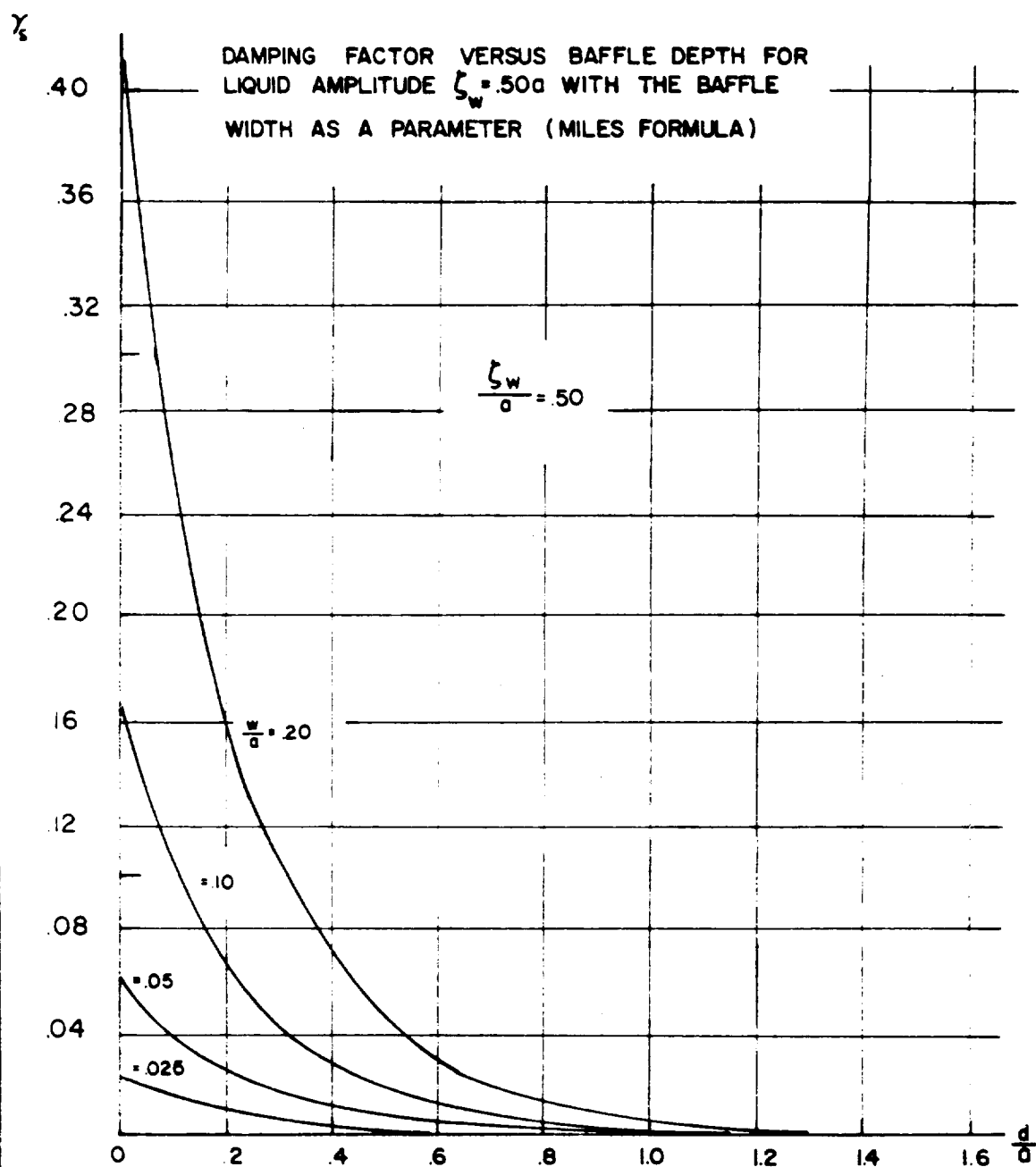


Figure 5-21

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=0.025$  WITH LIQUID AMPLITUDE AS A PARAMETER  
(MILES FORMULA)

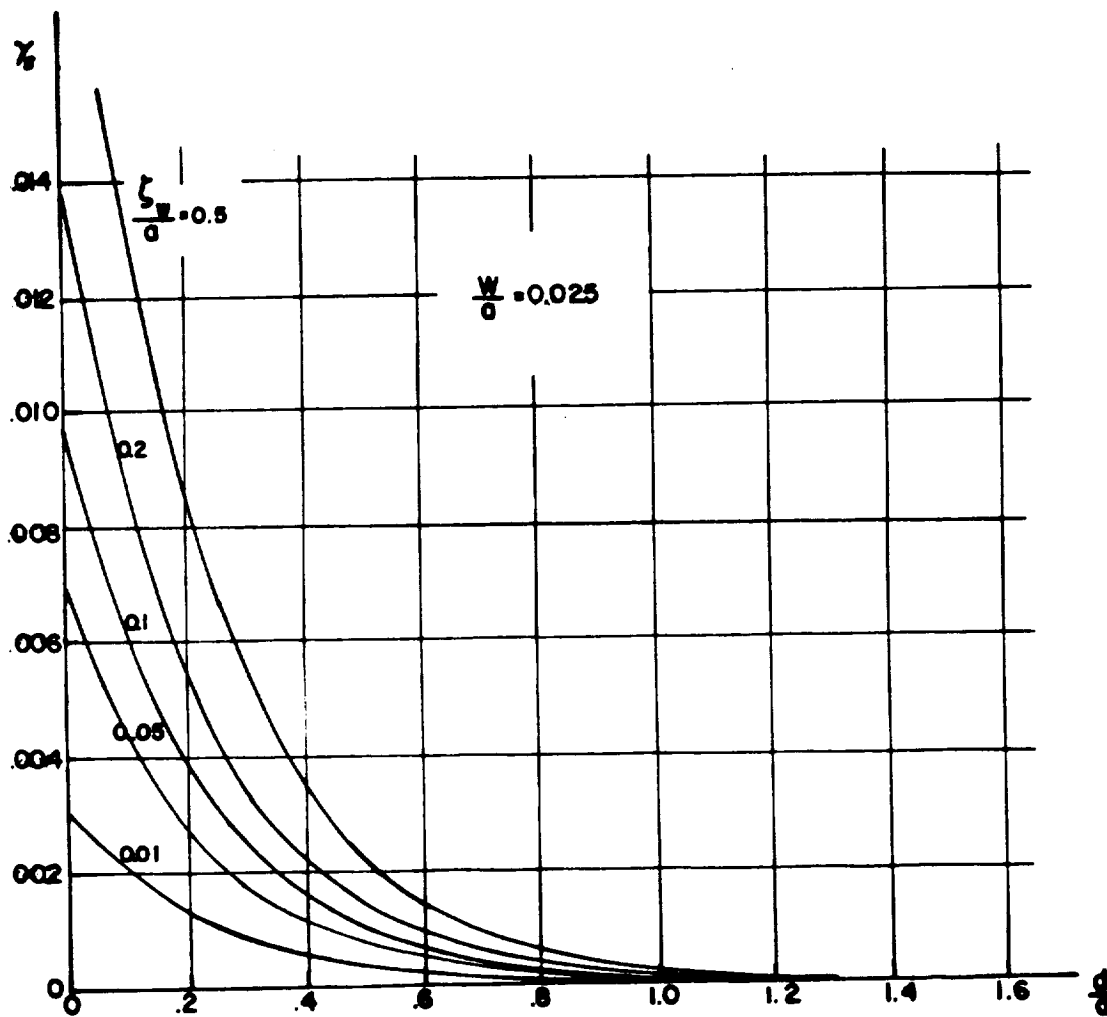


Figure 5-22

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=0.05a$  WITH LIQUID AMPLITUDE AS A PARAMETER  
(MILES FORMULA)

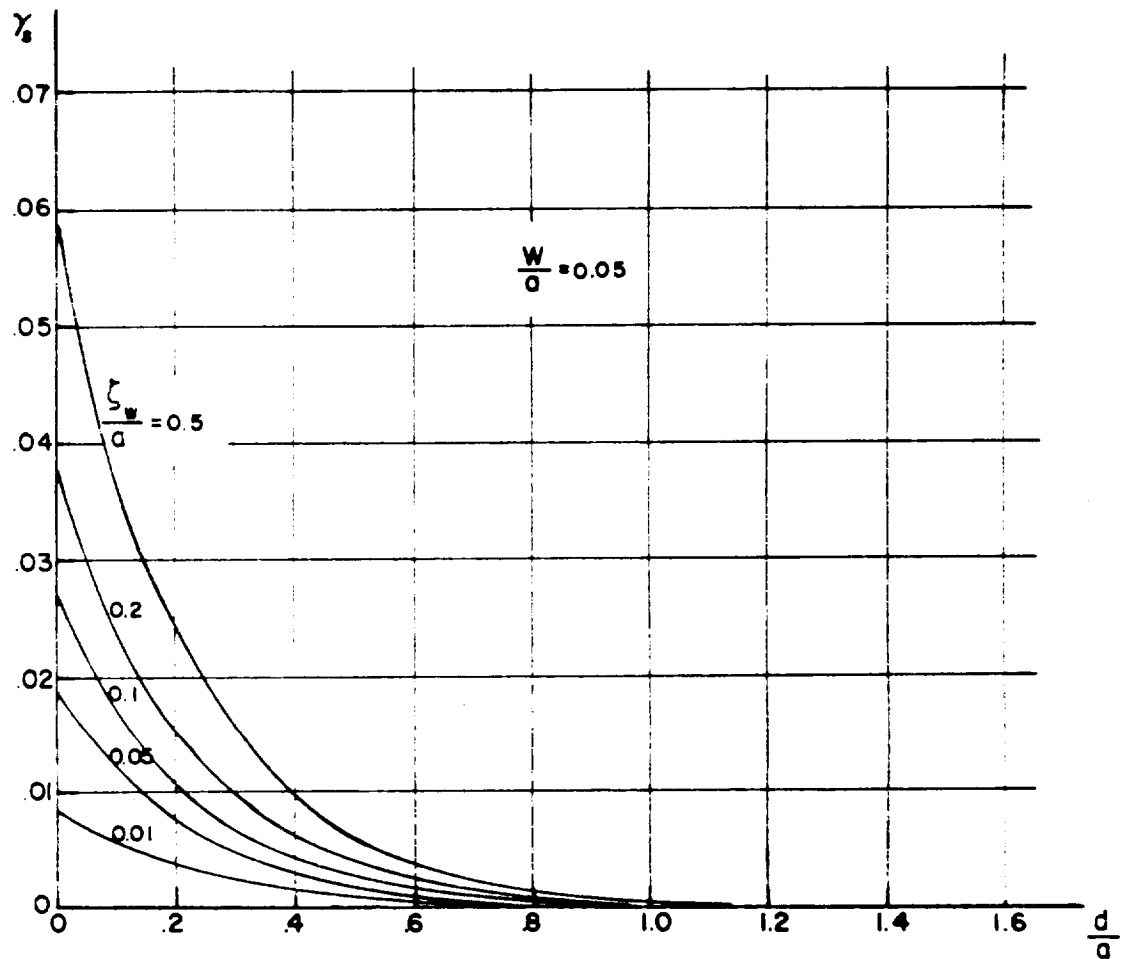


Figure 5-23

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=0.10a$  WITH LIQUID AMPLITUDE AS A PARAMETER  
(MILES FORMULA)

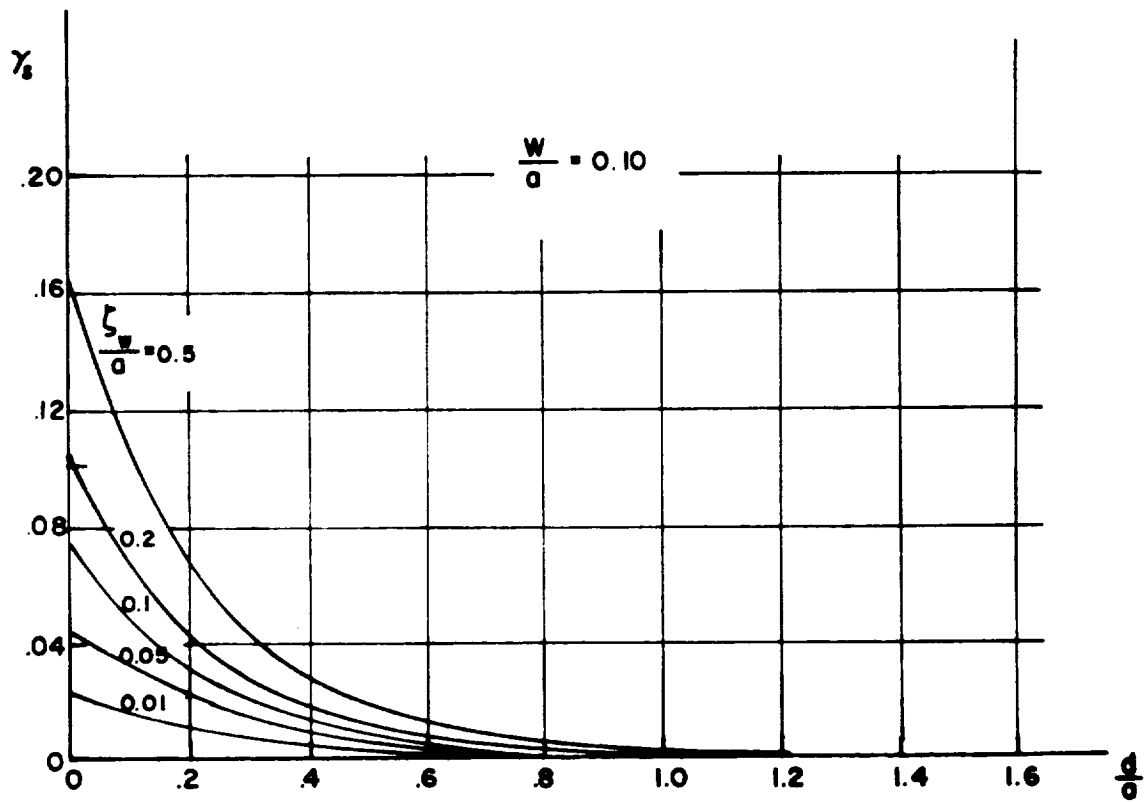


Figure 5-24



DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=0.20a$  WITH LIQUID AMPLITUDE AS A PARAMETER  
 (MILES FORMULA)

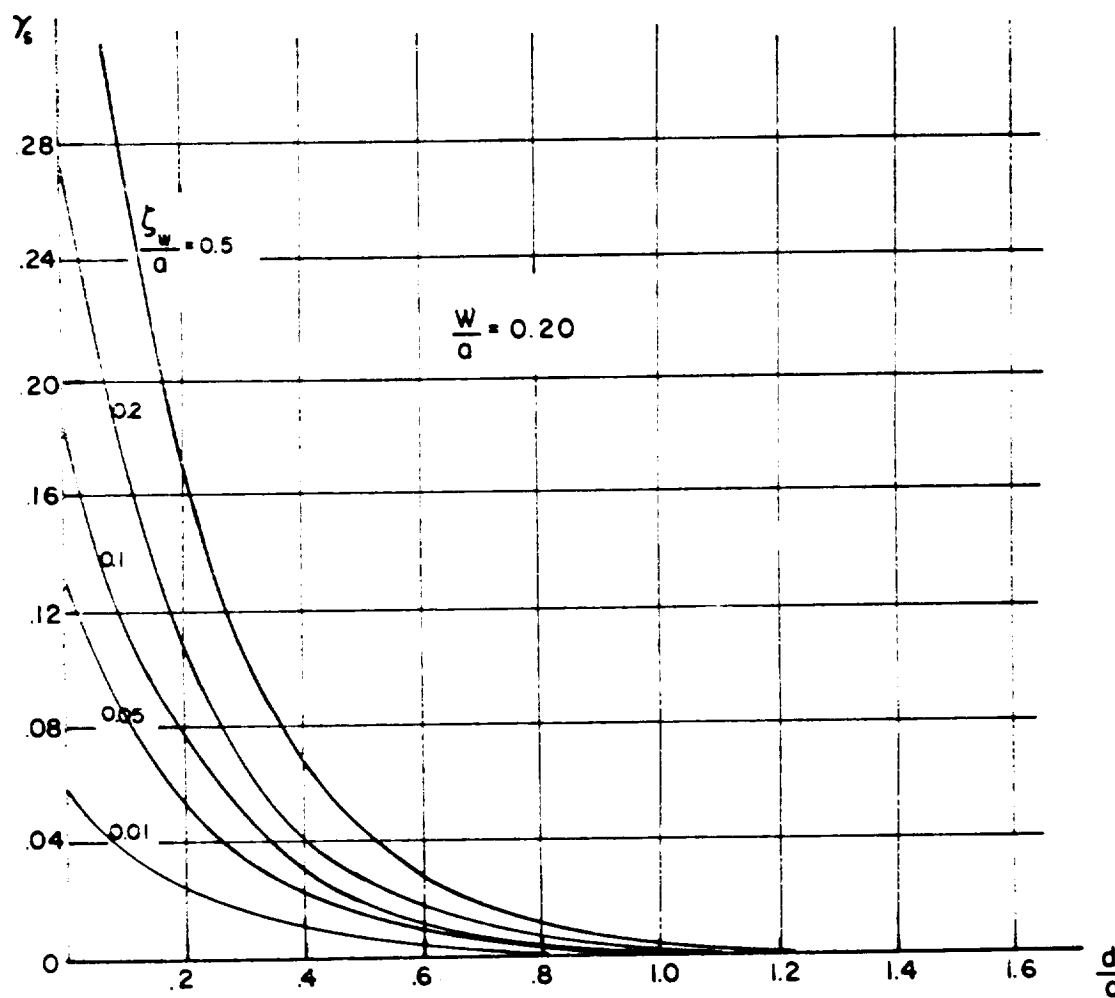


Figure 5-25

DAMPING FACTOR VERSUS SURFACE AMPLITUDE FOR BAFFLE  
WIDTH  $W = .05a$  WITH BAFFLE LOCATION AS A PARAMETER  
(MILES FORMULA)

$$\frac{W}{a} = .05$$

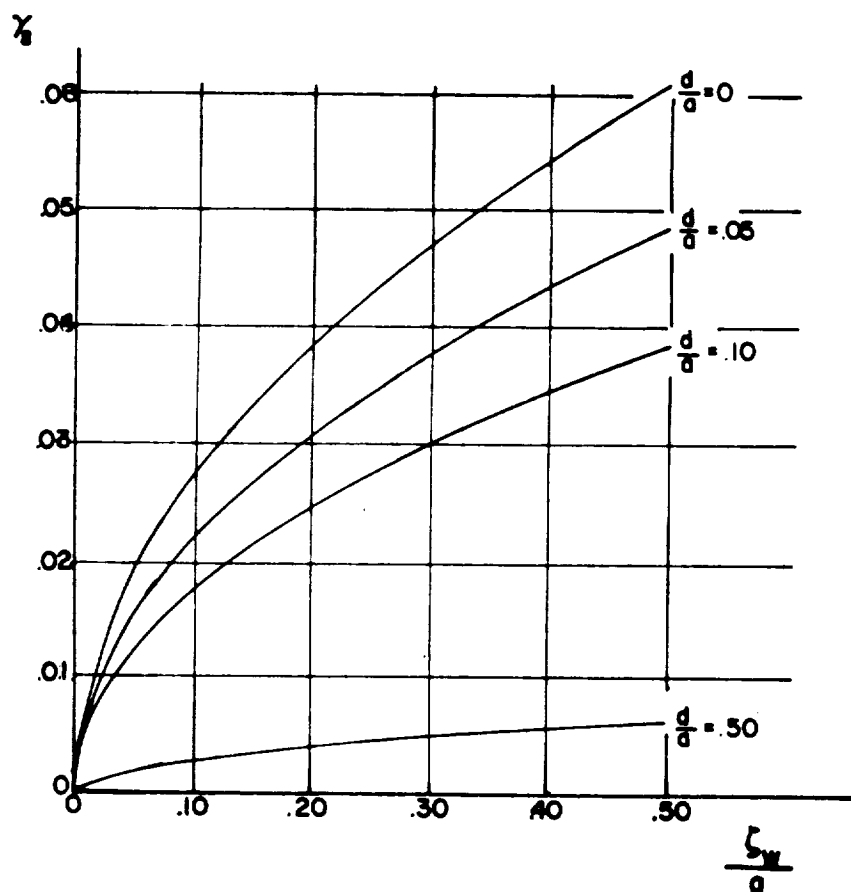


Figure 5-26

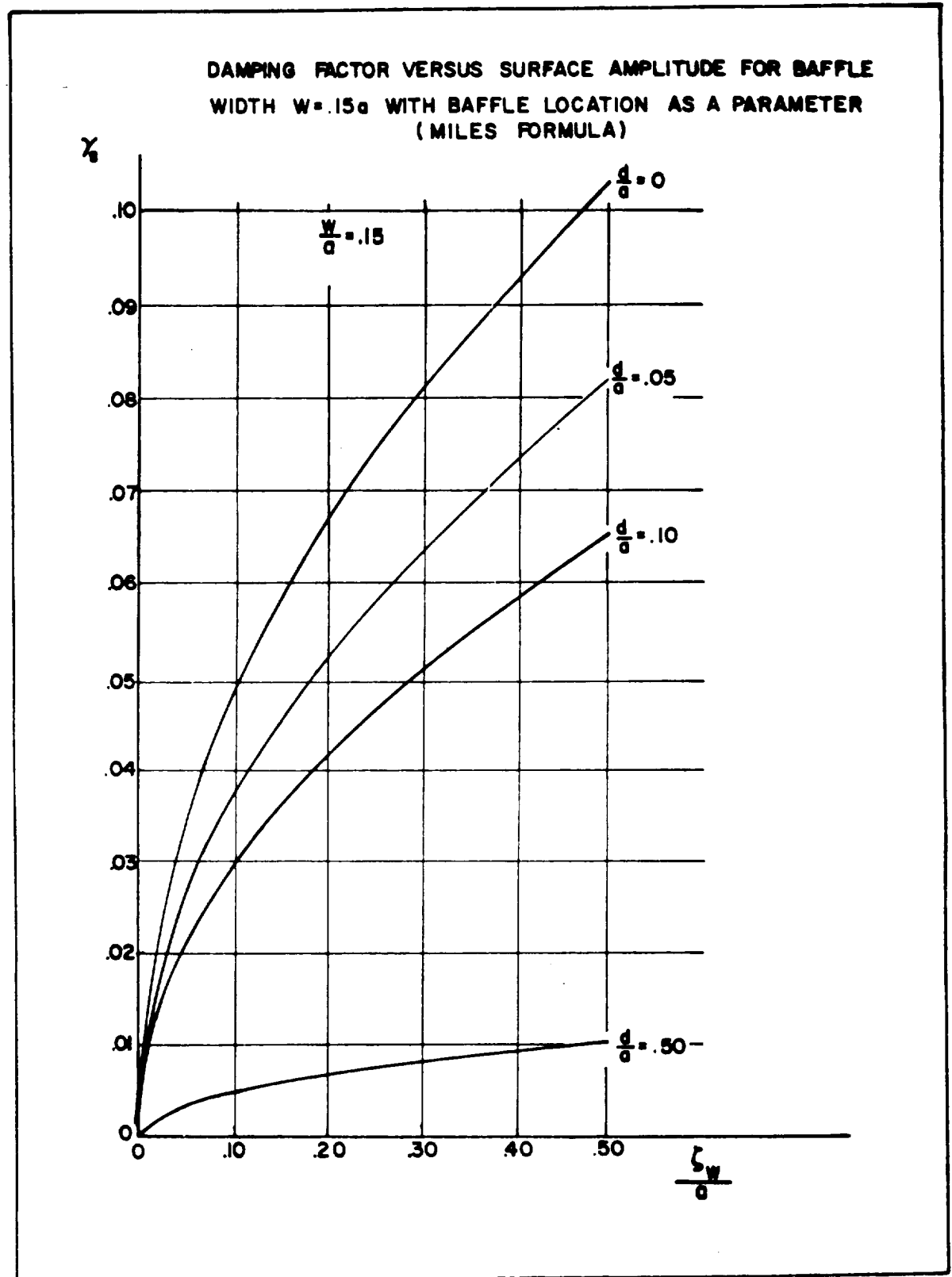


Figure 5-27

DAMPING FACTOR VERSUS BAFFLE WIDTH FOR  
 $\zeta_w = .10a$  WITH LOCATION OF BAFFLE AS A  
 PARAMETER (MILES FORMULA)

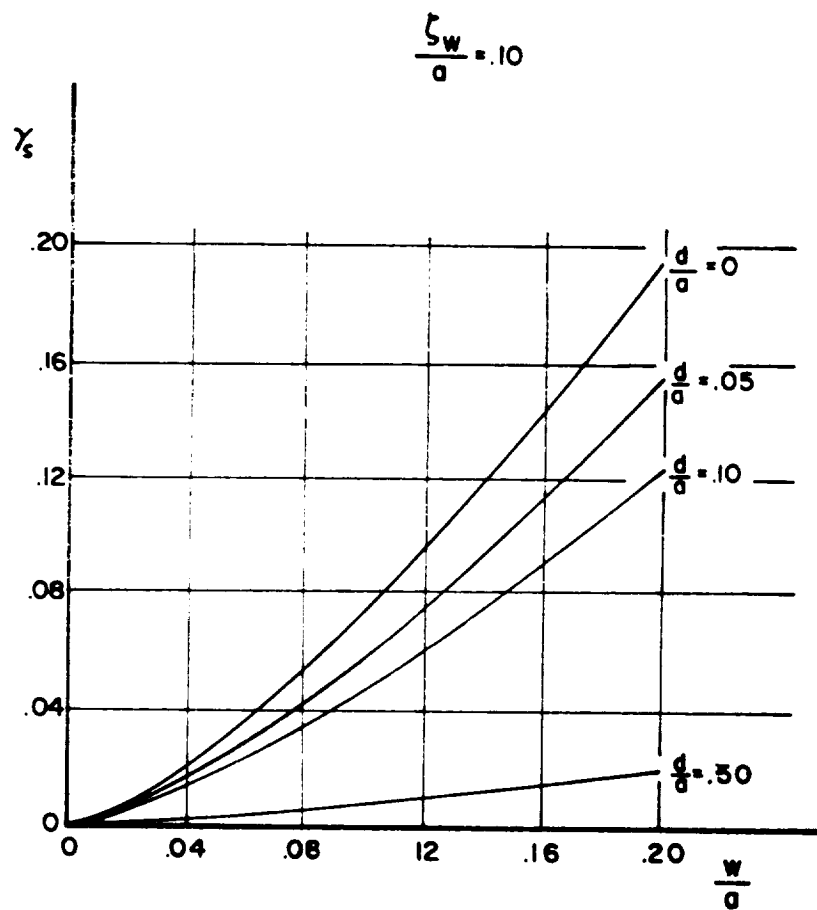


Figure 5-28

DAMPING FACTOR VERSUS BAFFLE WIDTH FOR  
 $\zeta_w = .30a$  WITH LOCATION OF BAFFLE AS A  
 PARAMETER (MILES FORMULA)

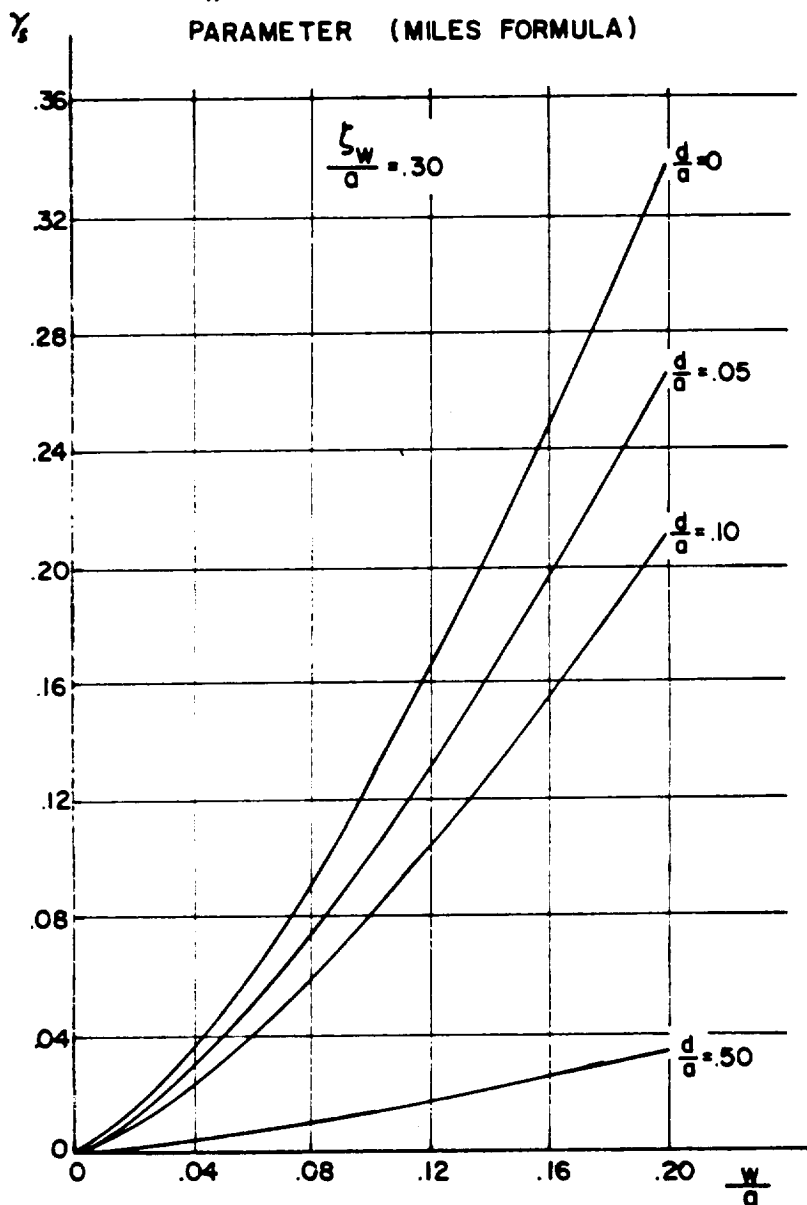


Figure 5-29

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2.2 O'Neill's Extension (84). Within the scatter of experimental data, O'Neill showed that either of the following equations can be used for reasonable values of  $\frac{\zeta_1}{a}$  and  $\bar{F} = \frac{F}{\rho g a^3}$ , for any ring submergence  $\frac{d}{a} \geq 0$  and for ring widths corresponding to  $\alpha \leq 0.25$

$$\begin{aligned} \gamma &= 2.83 e^{-4.60d/a} \alpha^{3/2} \left( \frac{\zeta_1}{a} \right)^{1/2} & (\text{Miles}) \\ \gamma &= 2.16 e^{-4.60d/a} \alpha^{3/2} \left( \frac{F}{\rho g a^3} \right)^{1/2} & (5-2) \end{aligned}$$

Equation (5-2) gives the damping factor in terms of the lateral liquid force  $F$  which is easier to measure than the wave amplitude  $\zeta_1$ .

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2.3 Bauer's Extension (31). During sloshing, the baffles and the oscillating fluid are not in continuous contact with each other. There are parts of the oscillation cycle during which the fluid does not uniformly wet the entire surface of the baffles. For this reason, Miles (28) expression, Equation (5-1), gives a higher damping effect than that experimentally observed. To account for the deviation, an effective baffle area concept was herewith successfully employed. This effective baffle area is a function of the width  $w$  of the baffle, its location  $d$  below the liquid free surface, and the maximum amplitude of sloshing  $\zeta_1$ . The theoretical formula obtained agrees with experimental results.

In Equation (5-1), the baffle area ratio blocking the cross sectional area is defined as

$$\alpha = \frac{w}{a} \left[ 2 - \frac{w}{a} \right]$$

if the baffle is completely submerged during a slosh cycle. If the baffle is completely out of the liquid during a certain time of the slosh cycle, the effective baffle area is

$$\begin{aligned} \bar{\alpha} = & \frac{\frac{w}{a} (2 - \frac{w}{a}) - \frac{1}{2} (1 - \frac{d/a}{\zeta_1/a}) + \frac{2}{\pi} \frac{d/a}{\zeta_1/a} \ln \left[ \frac{\zeta_1/a + (\zeta_1/a)^2 - (d/a)^2}{d/a} \right]}{-----} \\ & - \frac{1}{\pi} \frac{d/a}{(\zeta_1/a)^2} \sqrt{(\zeta_1/a)^2 - (d/a)^2} + \frac{1}{\pi} \arcsin \left( \frac{d/a}{\zeta_1/a} \right) \\ & ----- \\ & - \frac{1}{2} \frac{d/a}{\zeta_1/a} + \frac{1}{2} (1 - \frac{w}{a})^2 \left\{ 1 - \frac{d/a}{(1 - \frac{w}{a}) \zeta_1/a} \right\} \\ & ----- \end{aligned}$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2.3 Bauer's Extension (31) (continued)

$$\begin{aligned}
& - \frac{2}{\pi} \frac{\frac{d}{a} (1 - \frac{w}{a})}{\zeta_1/a} \ln \left[ \frac{\zeta_1/a + \sqrt{(\zeta_1/a)^2 - (\frac{d/a}{1 - w/a})^2}}{\frac{d/a}{1 - w/a}} \right] \\
& + \frac{1}{\pi} \frac{\frac{d}{a} (1 - \frac{w}{a})}{(\zeta_1/a)^2} \sqrt{(\zeta_1/a)^2 - \frac{(d/a)^2}{(1 - w/a)^2}} \\
& - \frac{1}{\pi} (1 - \frac{w}{a})^2 \arcsin \left( \frac{d/a}{(1 - \frac{w}{a}) \zeta_1/a} \right) + \frac{1}{2} (1 - \frac{w}{a}) \frac{d/a}{\zeta_1/a} \quad (5-3)
\end{aligned}$$

If the baffle is completely submerged, i.e.,  $\frac{d}{a} \geq \frac{\zeta_1}{a}$ , then the value of  $\bar{a}$  is equal to the underlined term (Equation 5-3). For the case of the baffle being out of the liquid during the slosh cycle in such a fashion that the inner rim of the ring baffle is not out of the fluid at any time, i.e.,

$$\frac{\zeta_1}{a} (1 - \frac{w}{a}) \leq \frac{d}{a} \leq \frac{\zeta_1}{a}$$

the underlined dotted terms are added to obtain the effective baffle area. For the case in which part of the baffle gets out of the liquid, i.e.,

$$\frac{d}{a} < \frac{\zeta_1}{a} (1 - \frac{w}{a})$$

the total formula applies. Thus, Miles Equation for a single flat ring baffle becomes



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2.3 Bauer's Extension (31). (continued)

$$\gamma = 2.83 (\bar{\alpha})^{3/2} e^{-\frac{4.6d}{a}} (\zeta_1/a)^{1/2} \quad (5-4)$$

For a system of ring baffles, the damping factor can be obtained by superimposing the contribution of each baffle. The nth baffle at a location  $d + (n-1)D$  below the free fluid surface exhibits an effective baffle area of

$$\begin{aligned} \bar{\alpha}_n = & \frac{w}{a} \left(2 - \frac{w}{a}\right) - \frac{1}{2} \left\{ 1 - \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{\zeta_1/a} \right\} \\ & + \left(\frac{2}{\pi}\right) \left[ \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{\zeta_1/a} \right] \cdot \ln \left\{ \frac{\frac{\zeta_1}{a} + \sqrt{\frac{\zeta_1^2}{a^2} - \left(\frac{d}{a} + (n-1) \frac{D}{a}\right)^2}}{\frac{d}{a} + (n-1) \frac{D}{a}} \right\} \\ & - \left(\frac{1}{\pi}\right) \left[ \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{\zeta_1/a} \right] \cdot \sqrt{\left(\frac{\zeta_1}{a}\right)^2 - \left[\frac{d}{a} + (n-1) \frac{D}{a}\right]^2} \\ & + \left(\frac{1}{\pi}\right) \arcsin \left( \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{\zeta_1/a} \right) - \left(\frac{1}{2}\right) \left( \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{\zeta_1/a} \right) \\ & + \frac{1}{2} \left(1 - \frac{w}{a}\right)^2 \left\{ 1 - \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{(1 - \frac{w}{a})\zeta_1/a} \right\} \\ & - \frac{2}{\pi} \frac{\left[\frac{d}{a} + (n-1) \frac{D}{a}\right] \left(1 - \frac{w}{a}\right)}{\zeta_1/a} . \end{aligned}$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.3.2.3 Bauer's Extension (31) (continued)

$$\begin{aligned}
& \ln \left[ \frac{\frac{\zeta_1}{a} + \sqrt{\frac{\zeta_1}{a}^2 - \left[ \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{1 - \frac{w}{a}} \right]^2}}{\frac{\frac{d}{a} + (n-1) \frac{D}{a}}{1 - \frac{w}{a}}} \right] \\
& + \frac{1}{\pi} \frac{\left[ \frac{d}{a} + (n-1) \frac{D}{a} \right] (1 - \frac{w}{a})}{(\zeta_1/a)^2} \sqrt{\frac{\zeta_1}{a}^2 + \frac{\left[ \frac{d}{a} + (n-1) \frac{D}{a} \right]^2}{(1 - \frac{w}{a})^2}} \\
& - \frac{1}{\pi} (1 - \frac{w}{a})^2 \arcsin \left[ \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{(1 - \frac{w}{a}) \zeta_1/a} \right] \\
& + \frac{1}{2} (1 - \frac{w}{a}) \left[ \frac{\frac{d}{a} + (n-1) \frac{D}{a}}{\zeta_1/a} \right] \tag{5-5}
\end{aligned}$$

If

$$\frac{d}{a} + (n-1) \frac{D}{a} \geq \frac{\zeta_1}{a}$$

then

$$\bar{\alpha}_n = \alpha = \frac{w}{a} (2 - \frac{w}{a}) \tag{5-6}$$

because the complete baffle is submerged.

If

$$\frac{\zeta_1}{a} (1 - \frac{w}{a}) \leq \frac{d}{a} + (n-1) \frac{D}{a} \leq \frac{\zeta_1}{a}$$

then the underlined formula is added, while for the case

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2.3 Bauer's Extension (31) (continued)

$$\frac{d}{a} + (n-1) \frac{D}{a} < \frac{\zeta_1}{a} \left(1 - \frac{w}{a}\right)$$

the complete formula applies.

For a baffle located above the undisturbed liquid surface, the same formula can be applied approximately with a slight modification. If the baffle is completely out of the liquid, i.e.,

$$\frac{\zeta_1}{a} \leq \frac{d^*}{a},$$

where  $d^* = (D - d)$  is the distance of the baffle above the undisturbed liquid surface, the baffle area subjected to the fluid is  $\bar{\alpha}^* = 0$ . The value  $D$  is the distance between baffles. For

$$\frac{d^*}{a} < \frac{\zeta_1}{a} \leq \frac{d^*/a}{1 - \frac{w}{a}} \quad \text{and} \quad \frac{\zeta_1}{a} \geq \frac{d^*/a}{1 - \frac{w}{a}},$$

i.e., only a part of the baffle is subjected to the liquid during a slosh cycle, then the effective baffle area contributing to the damping is  $\bar{\alpha}^* = \alpha - \bar{\alpha}$ . For the  $m$ th baffle, this is

$$\bar{\alpha}_m^* = \frac{1}{2} \left(1 - \frac{m \frac{D}{a} - \frac{d}{a}}{\zeta_1/a}\right) - \frac{2}{\pi} \frac{\left(m \frac{D}{a} - \frac{d}{a}\right)}{\zeta_1/a} \ln \left[ \frac{\frac{\zeta_1}{a} + \sqrt{\left(\frac{\zeta_1}{a}\right)^2 - \left[m \frac{D}{a} - \frac{d}{a}\right]^2}}{m \frac{D}{a} - \frac{d}{a}} \right] +$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2.3 Bauer's Extension (31) (continued)

$$\begin{aligned}
& + \frac{1}{\pi} \frac{\left[ \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right]}{(\zeta_1/a)^2} \sqrt{\left( \frac{\zeta_1}{a} \right)^2 - \left( \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right)^2} - \frac{1}{\pi} \arcsin \left[ \frac{\left[ \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right]}{\zeta_1/a} \right] + \\
& + \frac{1}{2} \frac{\left( \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right)}{\zeta_1/a} - \frac{1}{2} \left( 1 - \frac{w}{a} \right)^2 \left\{ 1 - \frac{\left( \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right)}{\left( 1 - \frac{w}{a} \right) \zeta_1/a} \right\} + \\
& - \frac{2}{\pi} \frac{\left( \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right) \left( 1 - \frac{w}{a} \right)}{\zeta_1/a} \ln \frac{\frac{\zeta_1}{a} + \sqrt{\left( \frac{\zeta_1}{a} \right)^2 - \left( \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right)^2}}{\left( \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right) \left( 1 - \frac{w}{a} \right)} - \\
& - \frac{1}{\pi} \frac{\left[ \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right] \left( 1 - \frac{w}{a} \right)}{(\zeta_1/a)^2} \sqrt{(\zeta_1/a)^2 - \left[ \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right]^2} + \\
& + \frac{1}{\pi} \left( 1 - \frac{w}{a} \right)^2 \arcsin \left\{ \frac{\left( \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right)}{\left( 1 - \frac{w}{a} \right) \zeta_1/a} \right\} - \frac{1}{2} \left( 1 - \frac{w}{a} \right) \frac{\left[ \frac{m}{a} \frac{D}{a} - \frac{d}{a} \right]}{\zeta_1/a} .
\end{aligned}$$

(5-7)

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.2.3 Bauer's Extension (31) (continued)

If the value

$$m \frac{D}{a} - \frac{d}{a} \geq \frac{\zeta_1}{a},$$

then  $\bar{\alpha}_m^* = 0$ , while for the case where a partial part of the baffle is only subjected to the liquid during one slosh cycle, the effective area contributing to damping is presented by the underlined part of the formula. This means that this part is used for

$$\frac{\zeta_1}{a} (1 - \frac{w}{a}) \leq \frac{D}{a} - \frac{d}{a} < \frac{\zeta_1}{a}.$$

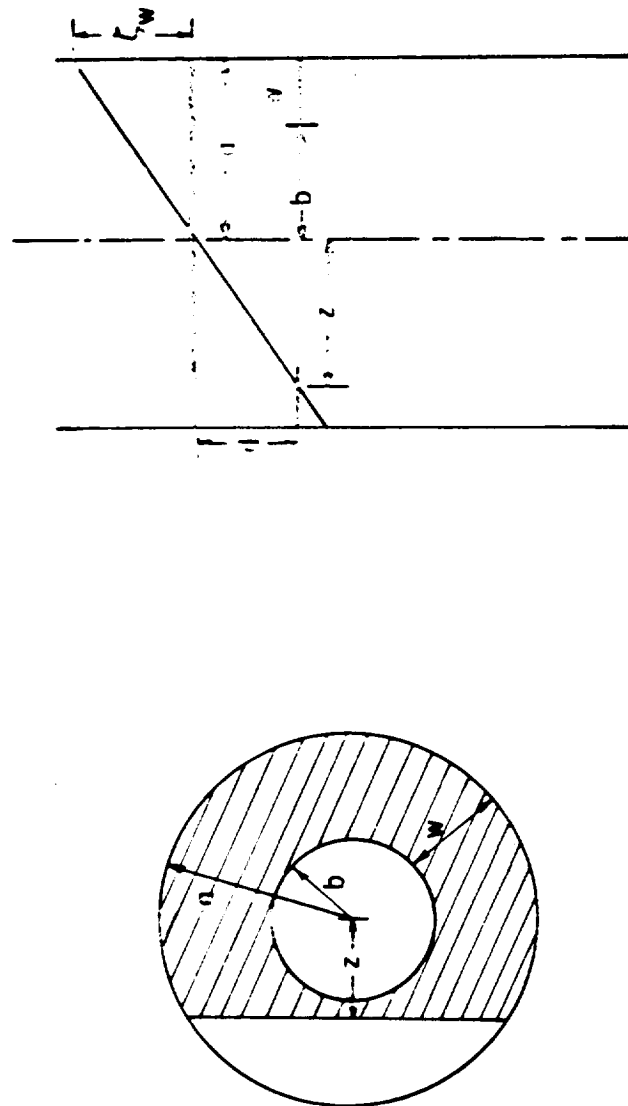
$$m \frac{D}{a} - \frac{d}{a} < \frac{\zeta_1}{a} (1 - \frac{w}{a}),$$

the total formula is applied for the effective area. Thus the total damping of  $N$  baffles of width  $w$  submerged into the undisturbed liquid and  $m$  baffles of the same width outside of the undisturbed fluid, all of which are apart from each other by the value  $D$ , is given by

$$\gamma_s = 2.83 \sqrt{\frac{\zeta_1}{a}} \left\{ \sum_{n=1}^N e^{-4.6 \left[ \frac{d}{a} + (n-1) \frac{D}{a} \right]} \cdot \bar{\alpha}_n^{3/2} + \sum_{m=1}^M e^{-4.6 \left( m \frac{D}{a} - \frac{d}{a} \right)} \bar{\alpha}_m^{*3/2} \right\}. \quad (5-8)$$

The result of this investigation can be seen in Figures 5-31 to 5-63.





ANNULAR RING BAFFLE PARTIALLY OUT OF LIQUID  
DURING SLOSH CYCLE

Figure 5-31

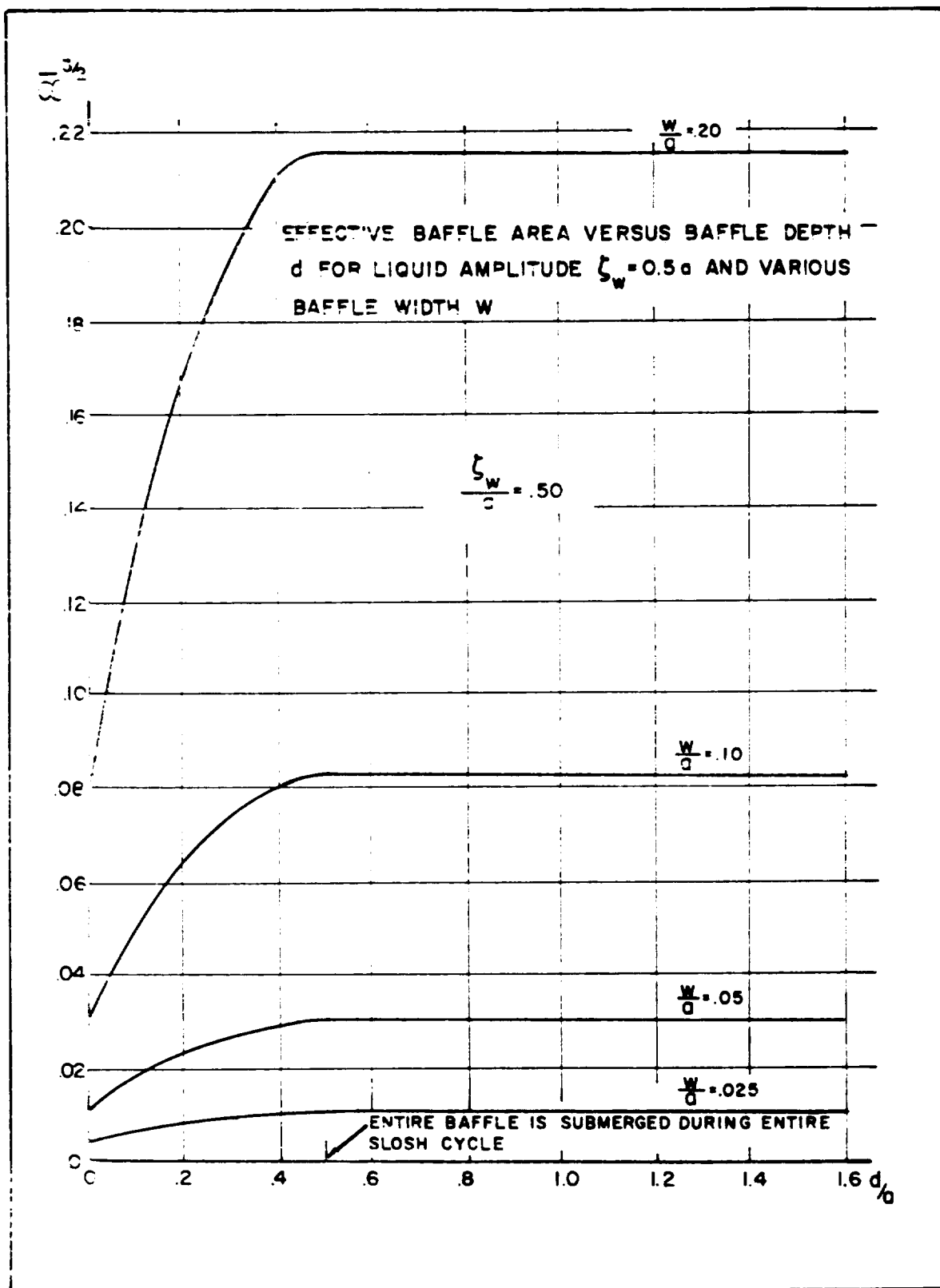


Figure 5-32



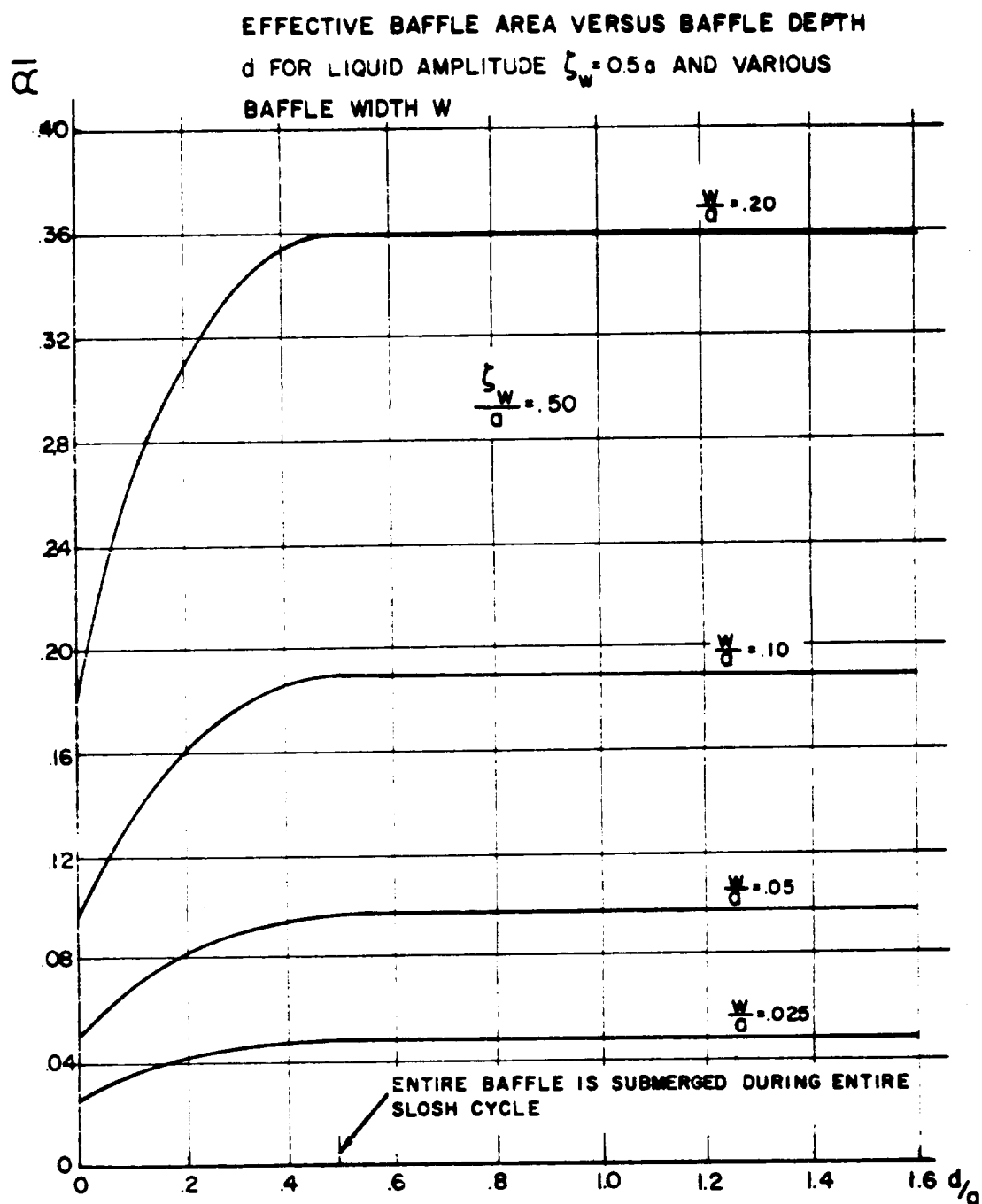


Figure 5-33

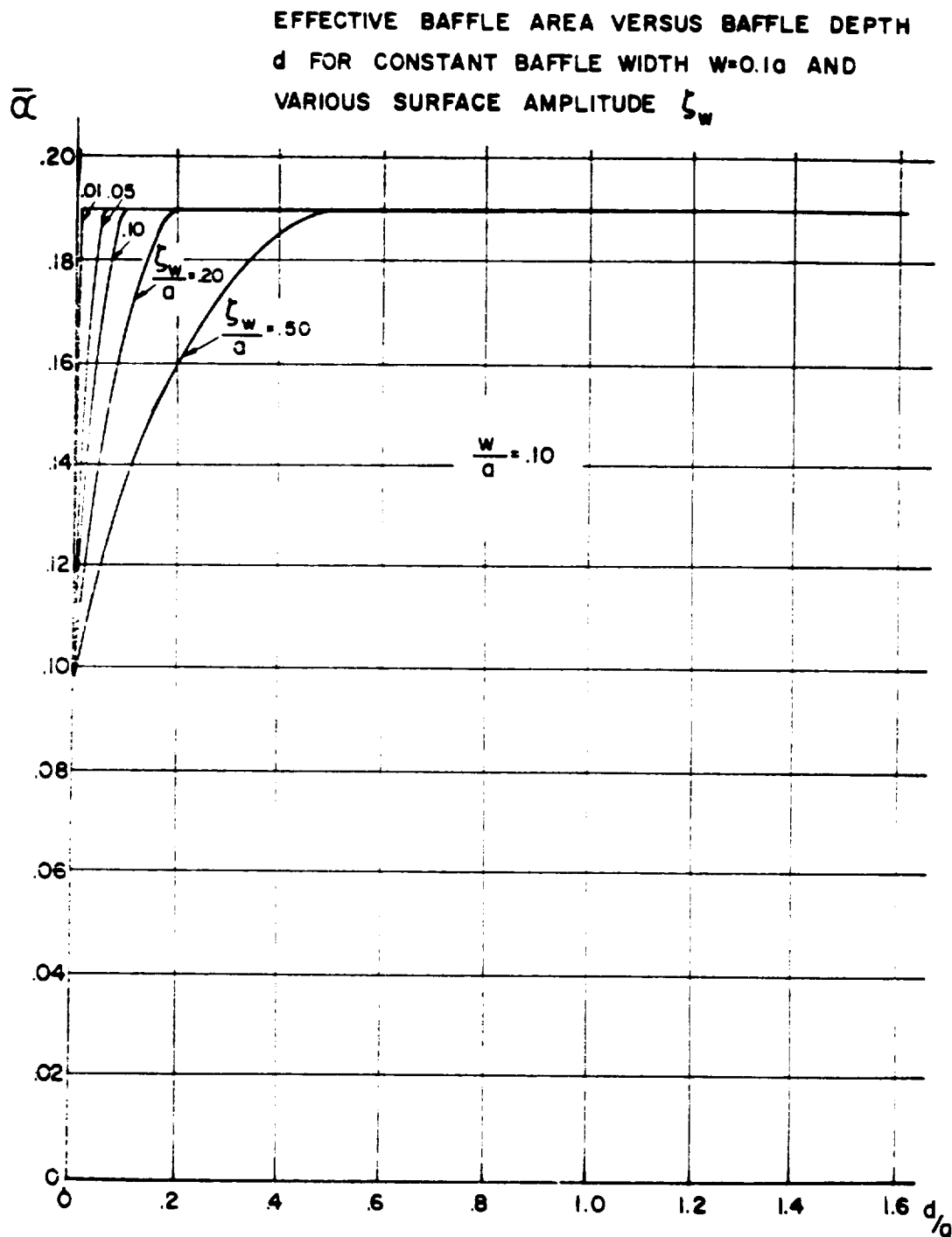


Figure 5-34

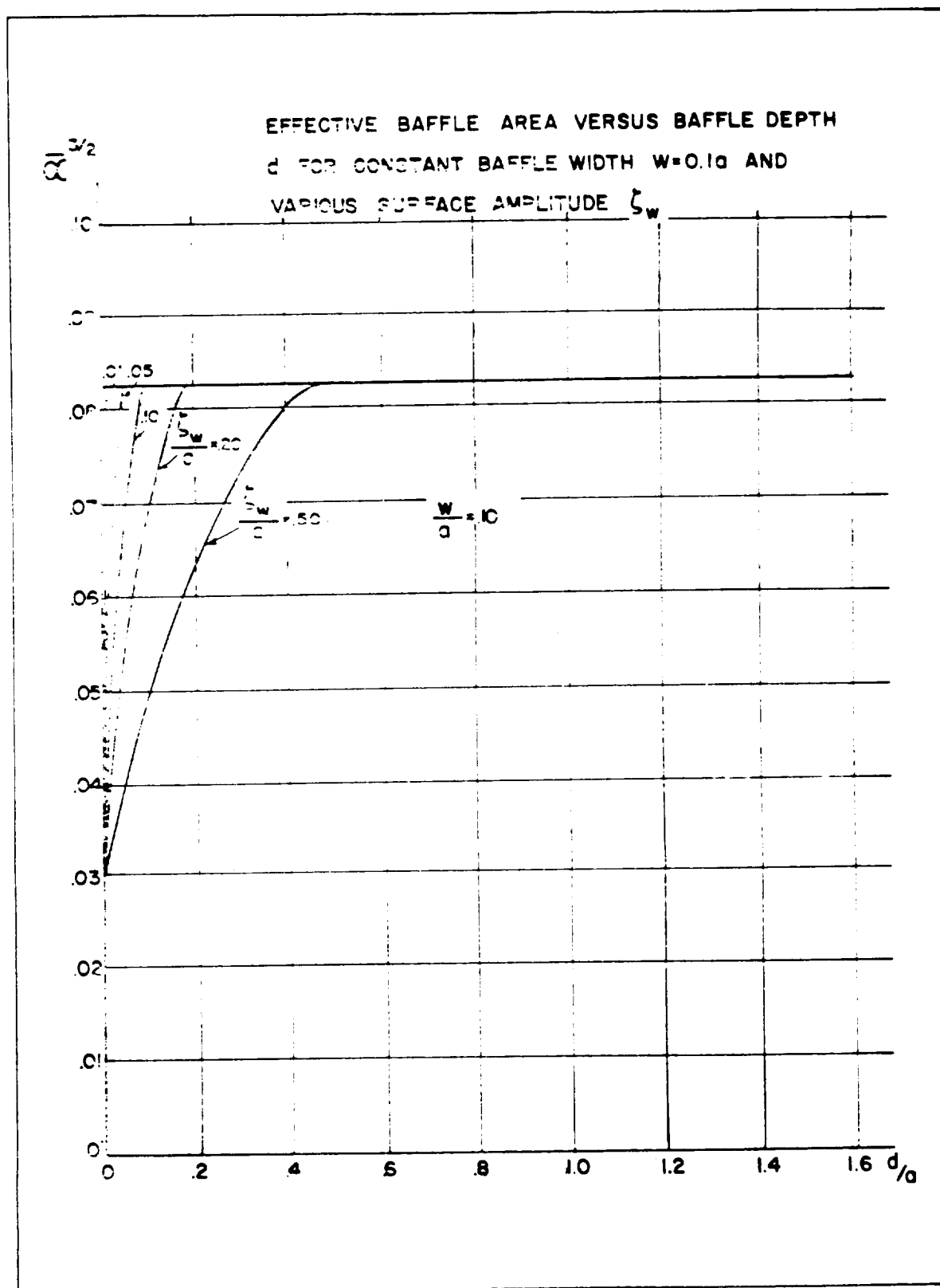


Figure 5-35

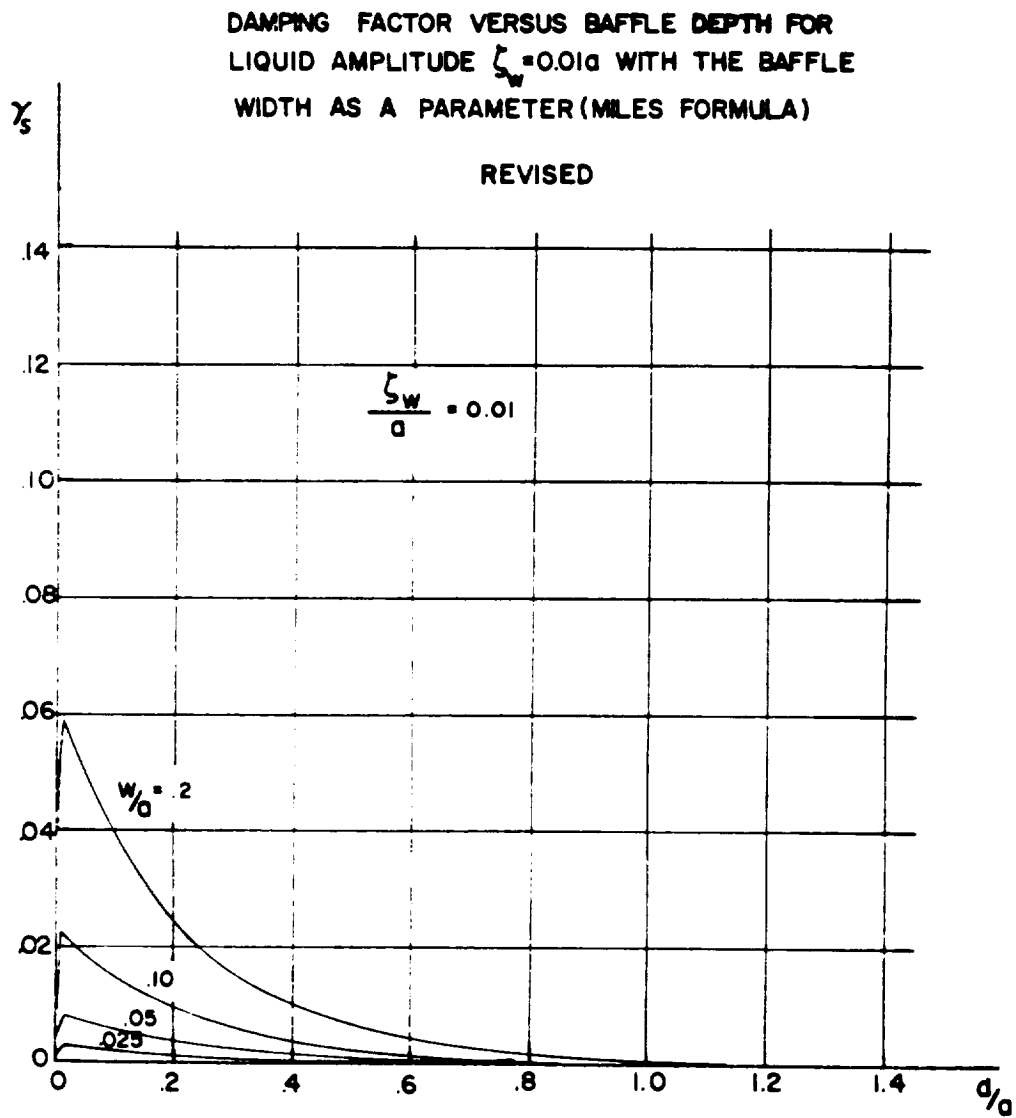


Figure 5-36

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR  
 LIQUID AMPLITUDE  $\zeta_w = 0.05d$  WITH THE BAFFLE  
 WIDTH AS A PARAMETER (MILES FORMULA)  
 REVISED

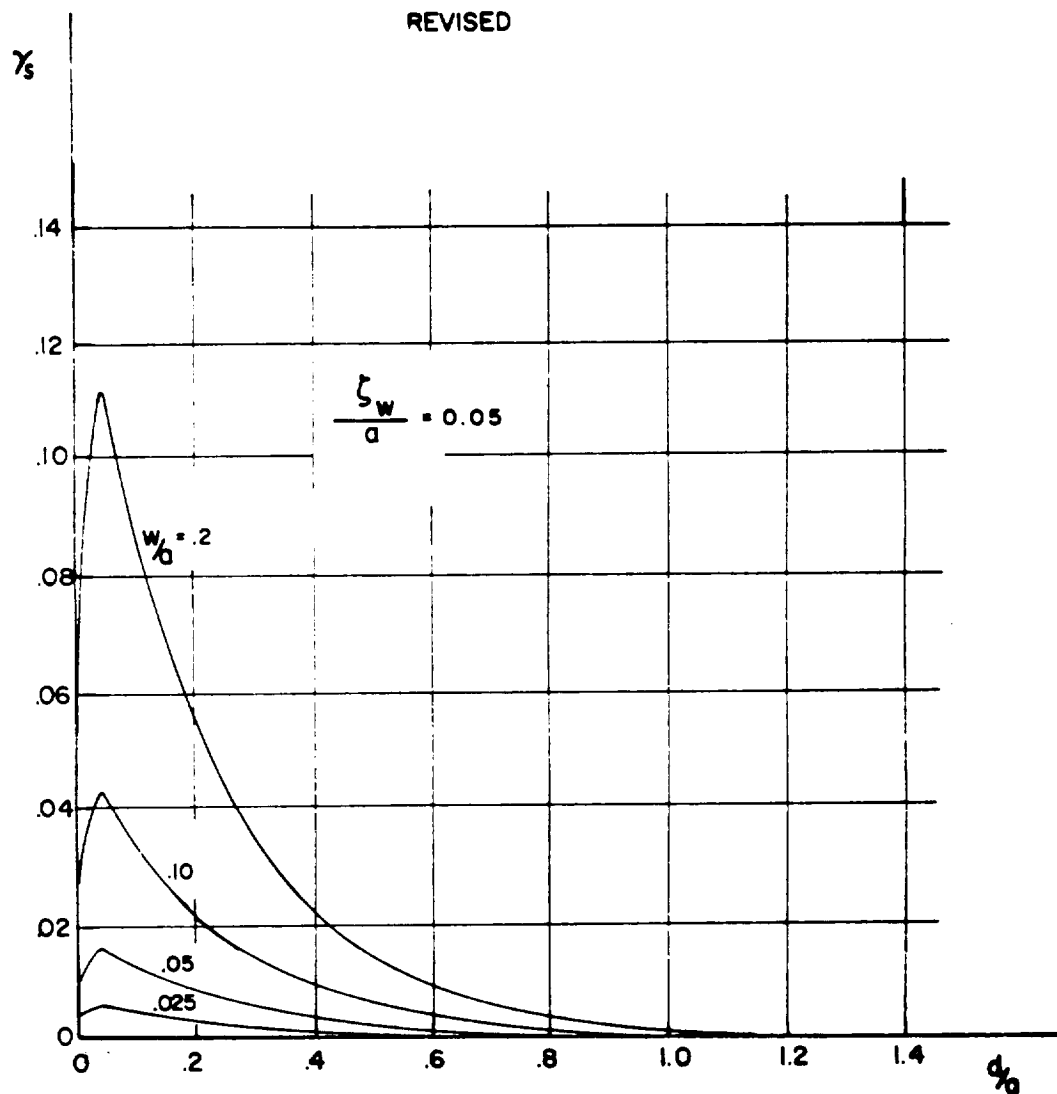


Figure 5-37

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR  
 LIQUID AMPLITUDE  $\zeta_w = 0.10a$  WITH THE BAFFLE  
 WIDTH AS A PARAMETER (MILES FORMULA)  
 REVISED

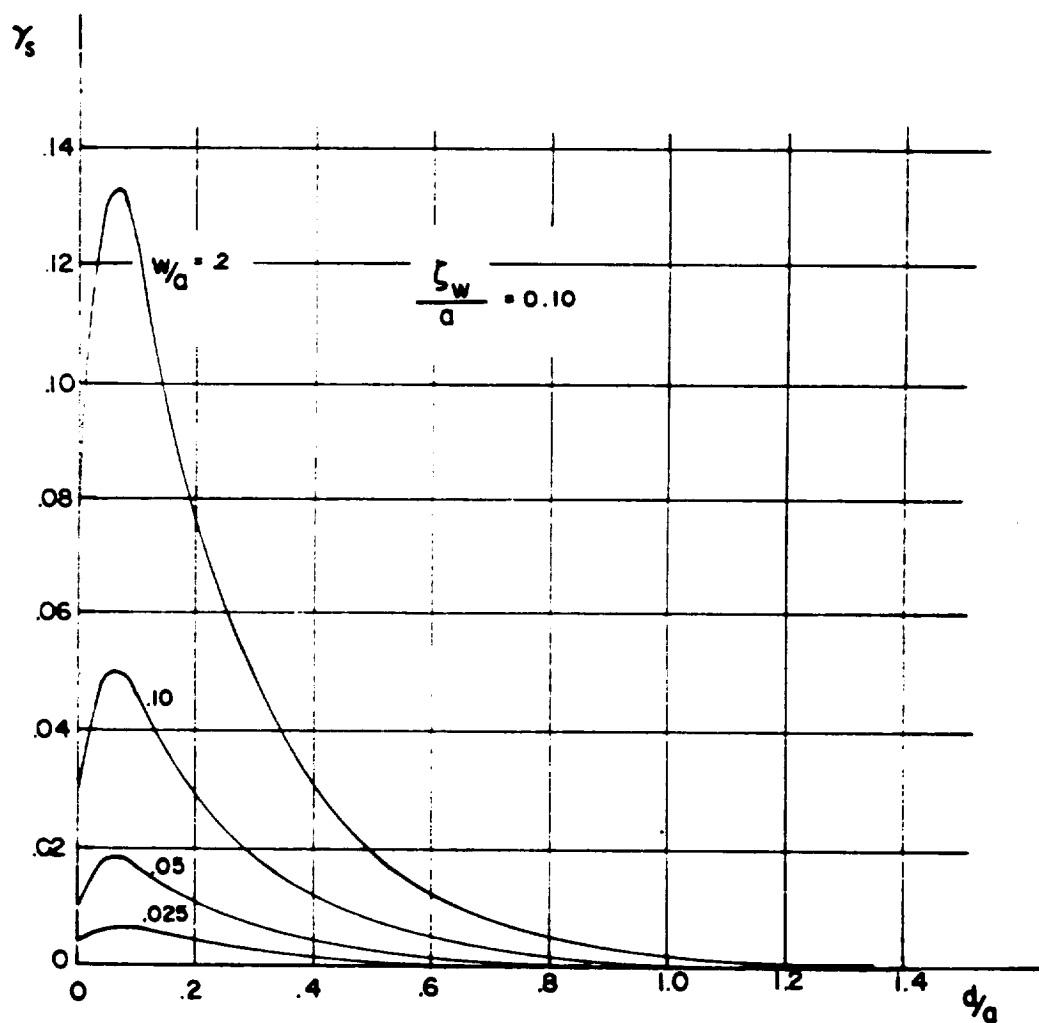


Figure 5-38

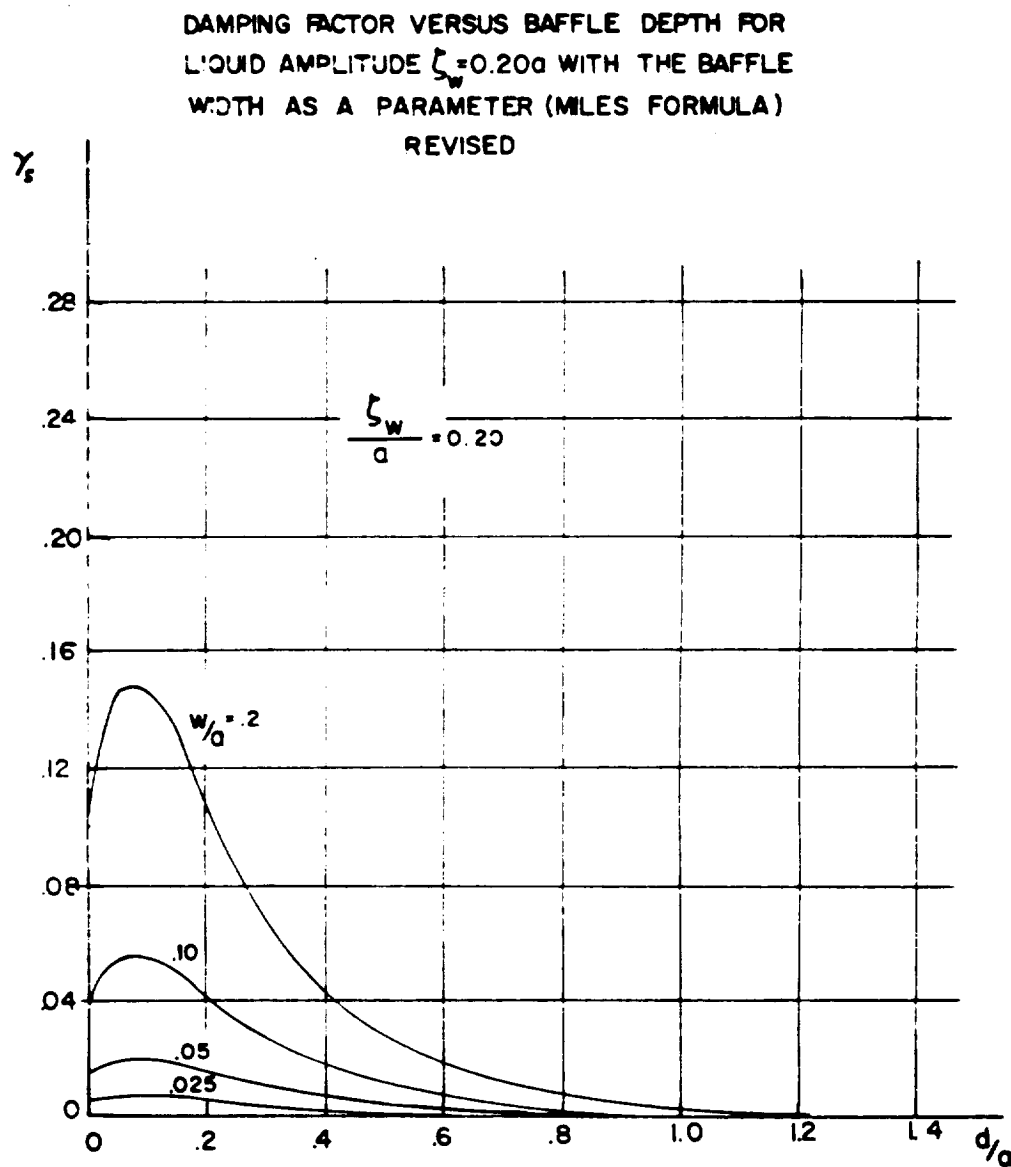


Figure 5-39

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR  
 LIQUID AMPLITUDE  $\zeta_w = 0.50a$  WITH THE BAFFLE  
 WIDTH AS A PARAMETER (MILES FORMULA)  
 REVISED

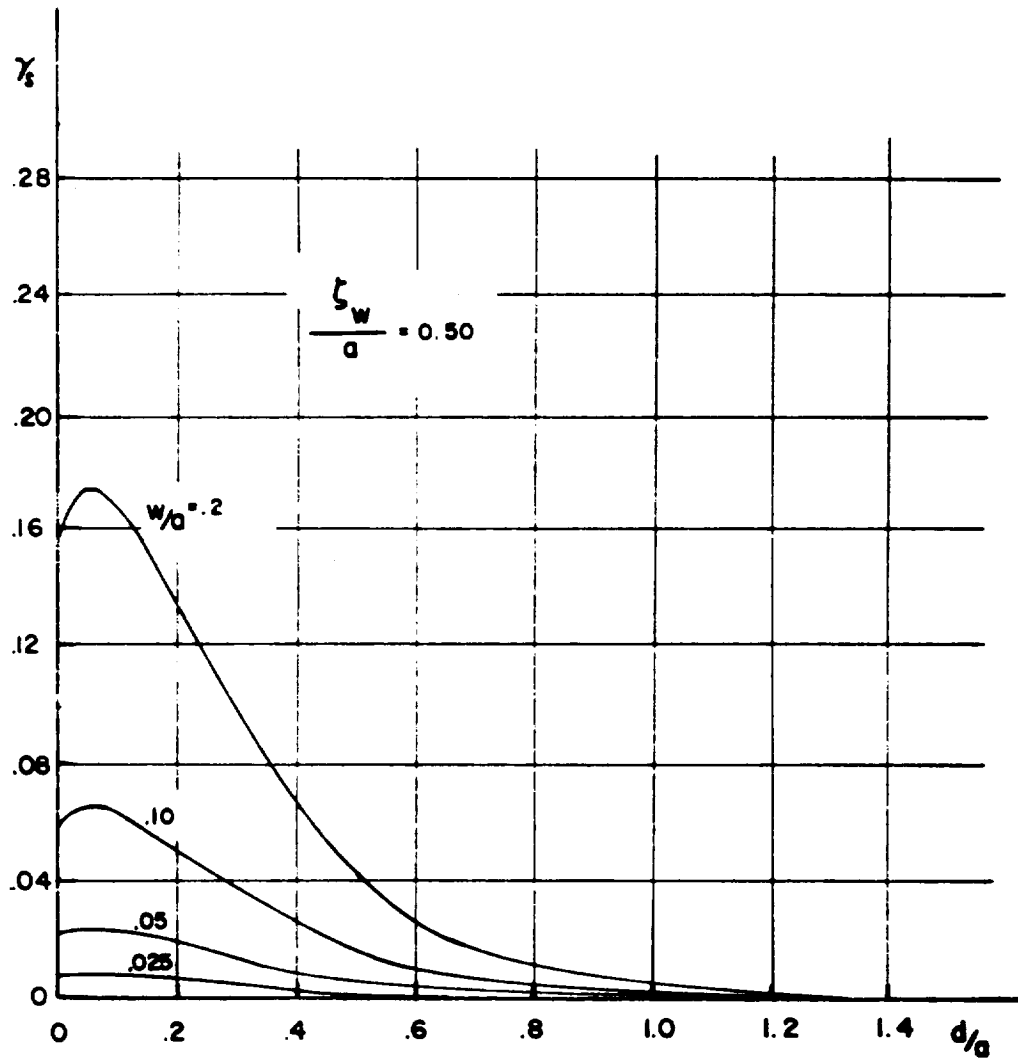


Figure 5-40



DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=0.025a$  WITH LIQUID AMPLITUDE AS A PARAMETER  
 (MILES FORMULA)

REVISED

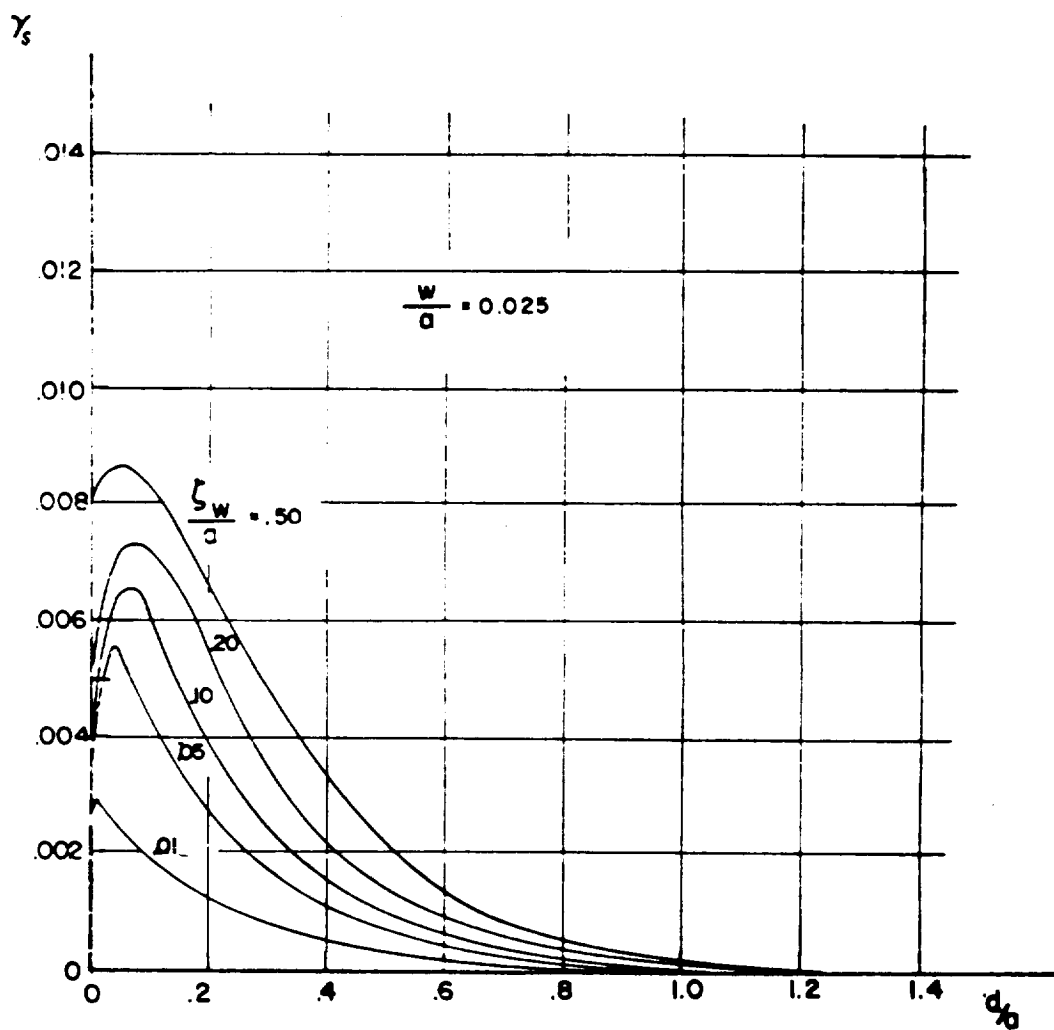


Figure 5-41

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=0.05d$  WITH LIQUID AMPLITUDE AS A PARAMETER  
 (MILES FORMULA)  
 REVISED

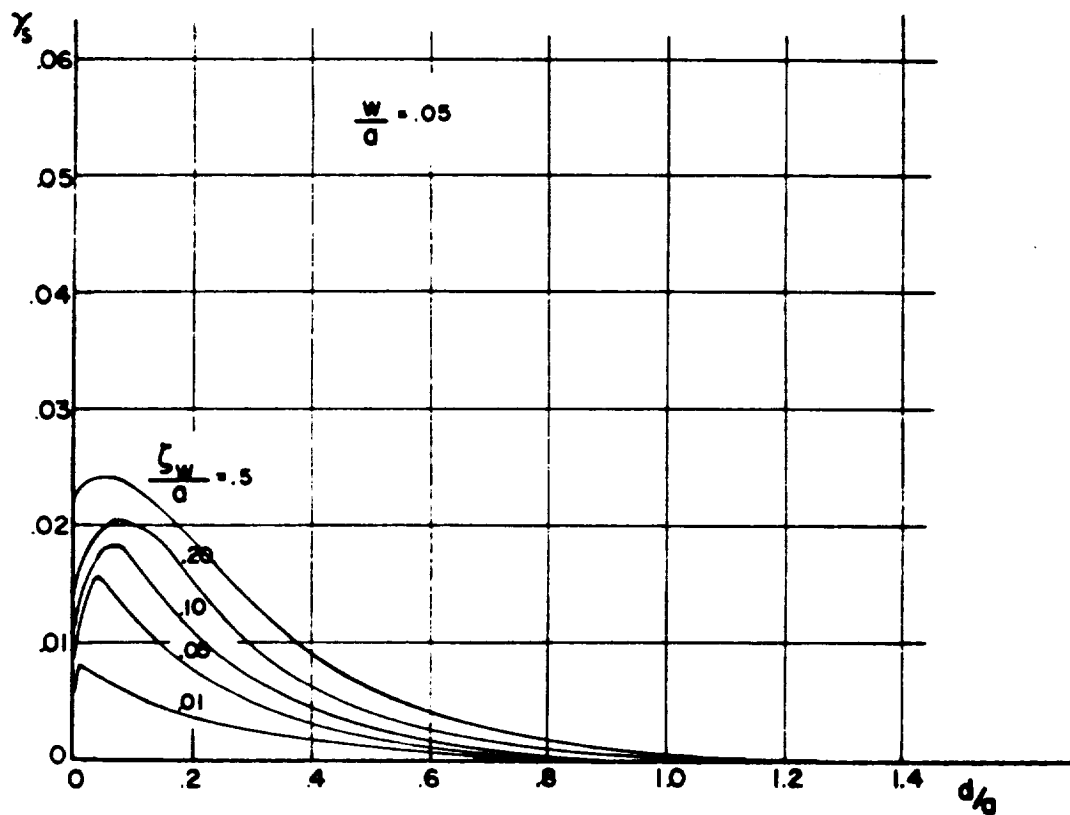


Figure 5-42

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=.10a$  WITH LIQUID AMPLITUDE AS A PARAMETER  
 (MILES FORMULA)

REVISED

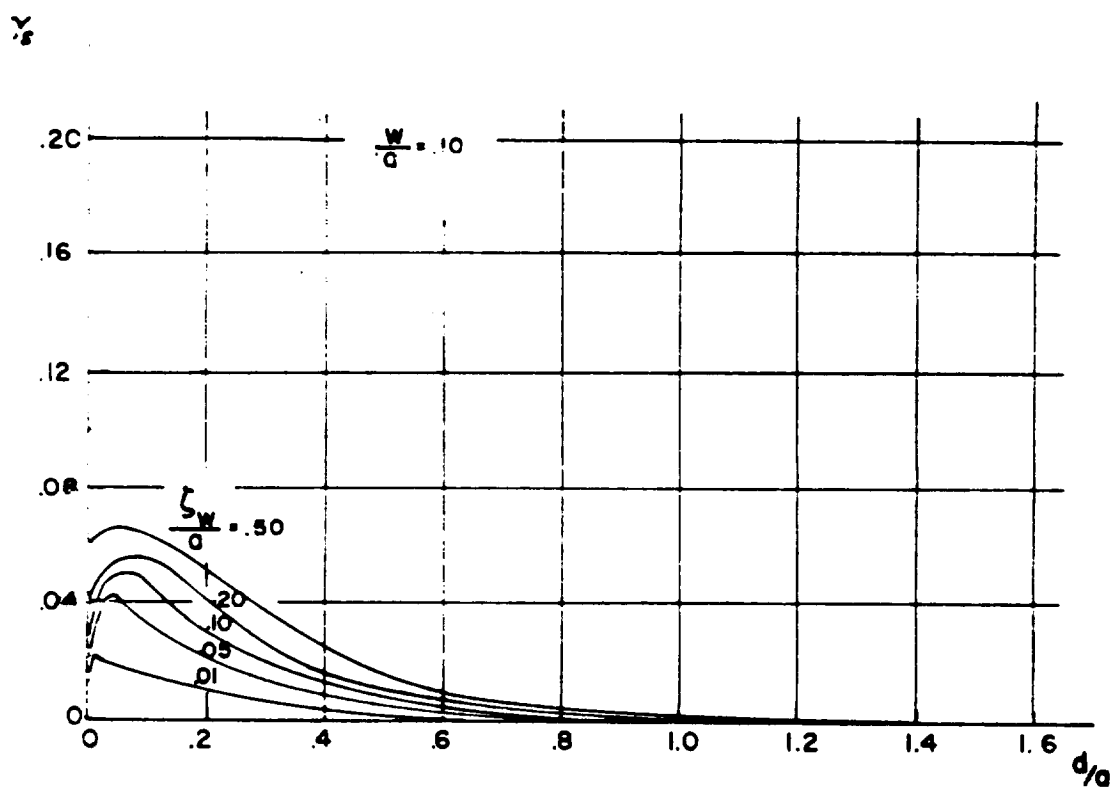


Figure 5-43

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR WIDTH  
 $W=0.20a$  WITH LIQUID AMPLITUDE AS A PARAMETER  
 (MILES FORMULA)  
 REVISED

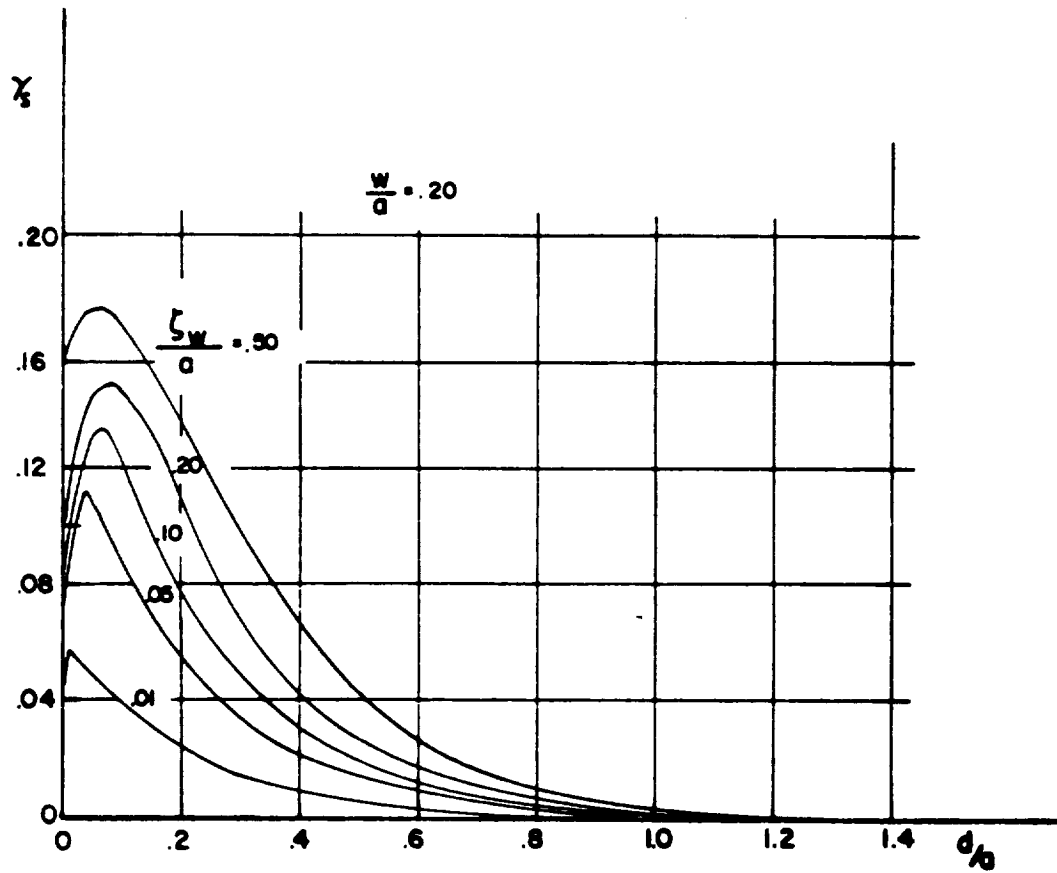


Figure 5-44

DAMPING FACTOR VERSUS SURFACE AMPLITUDE FOR  
 BAFFLE WIDTH  $W = .05a$  WITH BAFFLE LOCATION AS  
 A PARAMETER (MILES FORMULA)

REVISED

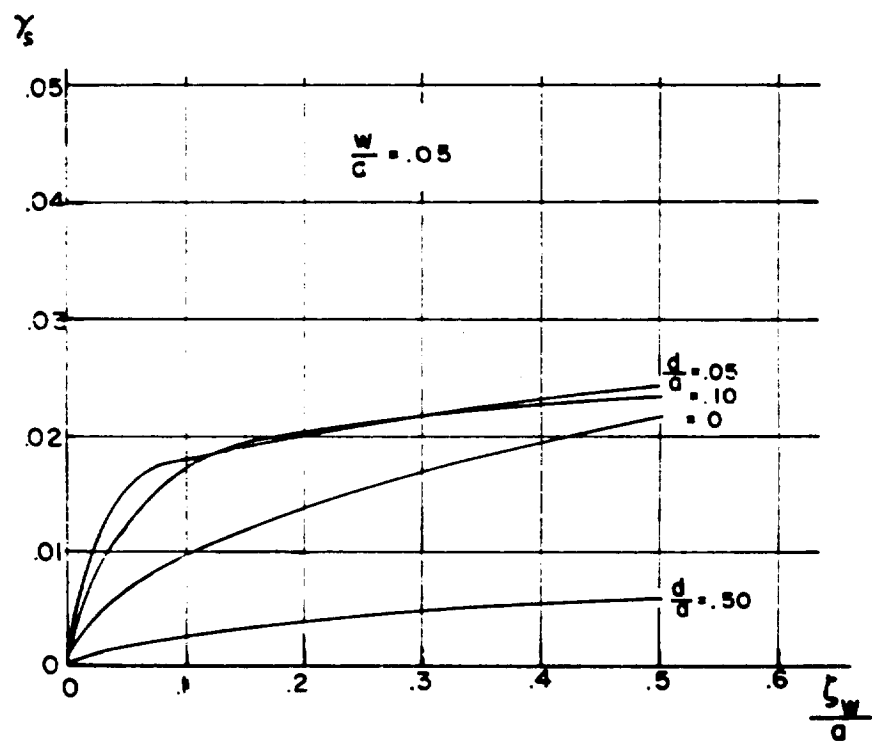


Figure 5-45

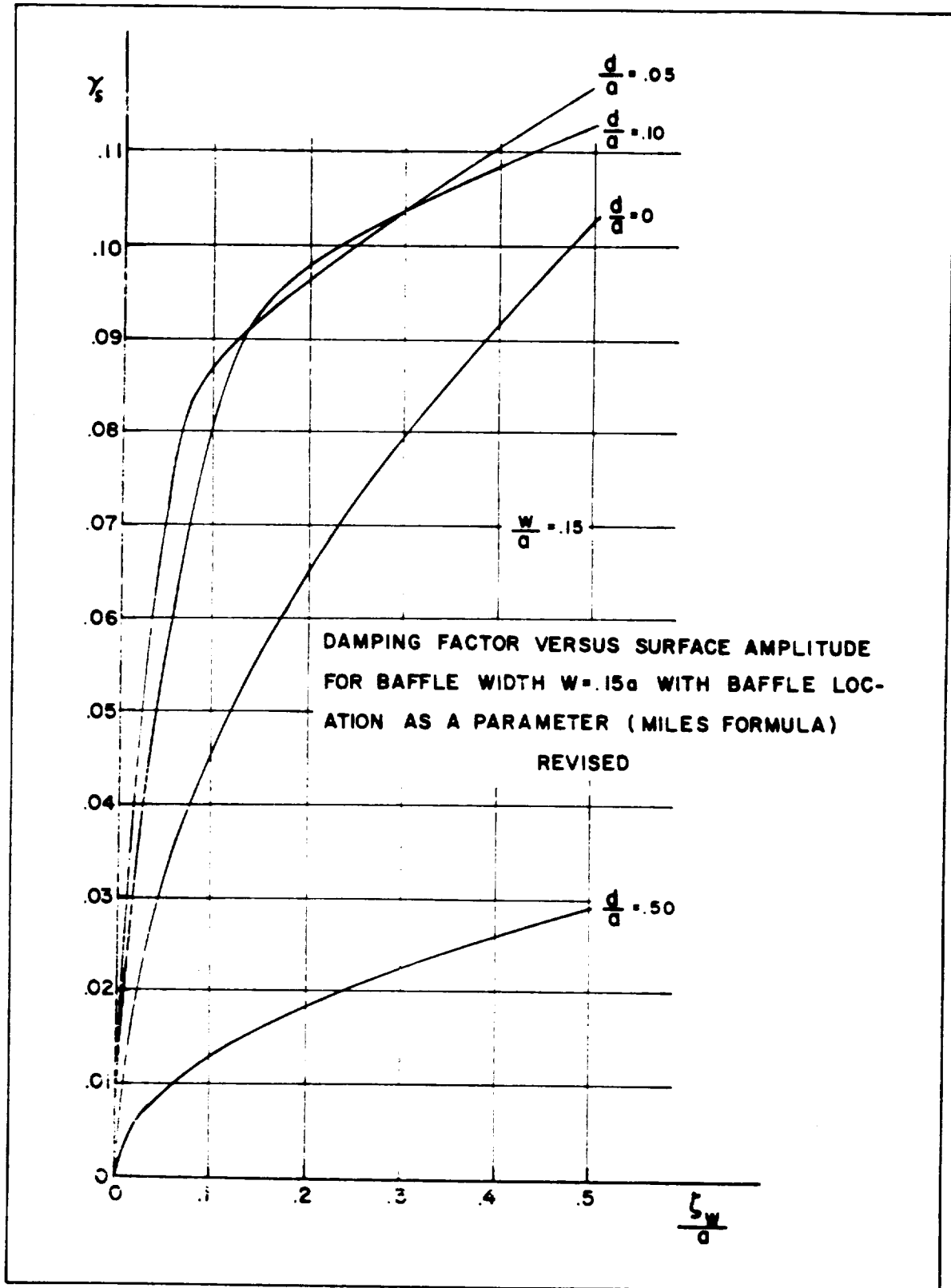


Figure 5-46

# DAMPING FACTOR VERSUS BAFFLE WIDTH FOR

$\zeta_w = .10$  WITH LOCATION OF BAFFLE AS A PARAMETER

(MILES FORMULA)

REVISED

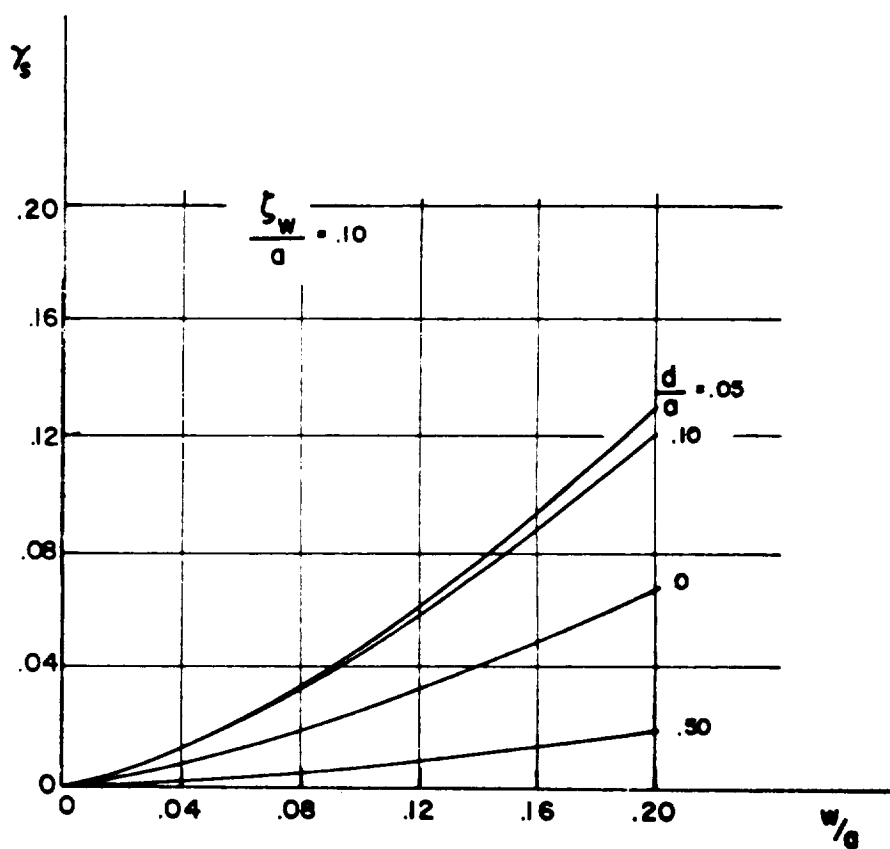


Figure 5-47

DAMPING FACTOR VERSUS BAFFLE WIDTH FOR  
 $\zeta_w = .30$  WITH LOCATION OF BAFFLE AS A PARAMETER  
 (MILES FORMULA)  
 REVISED

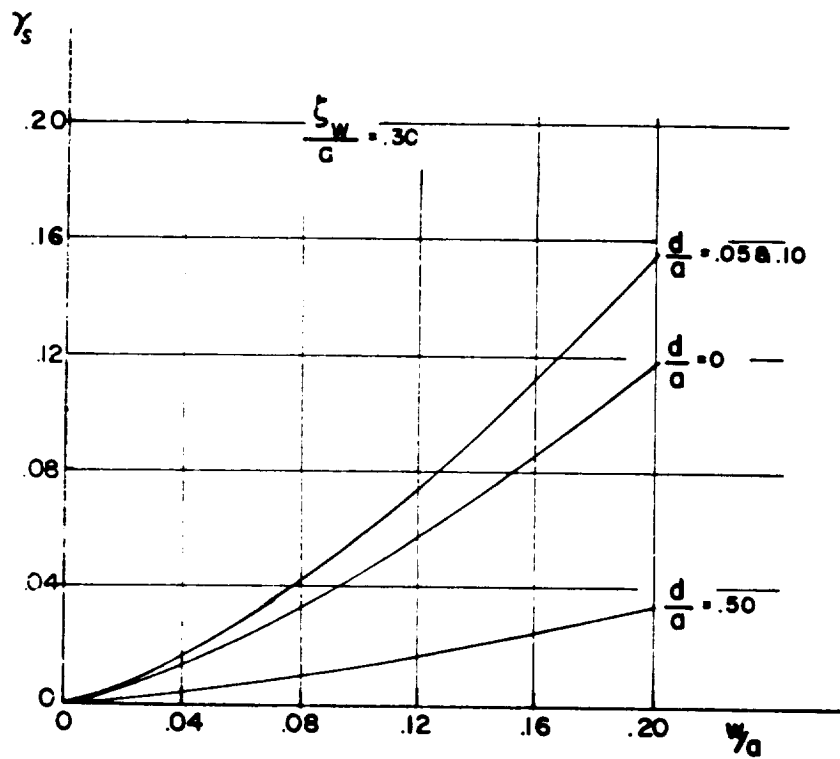


Figure 5-48



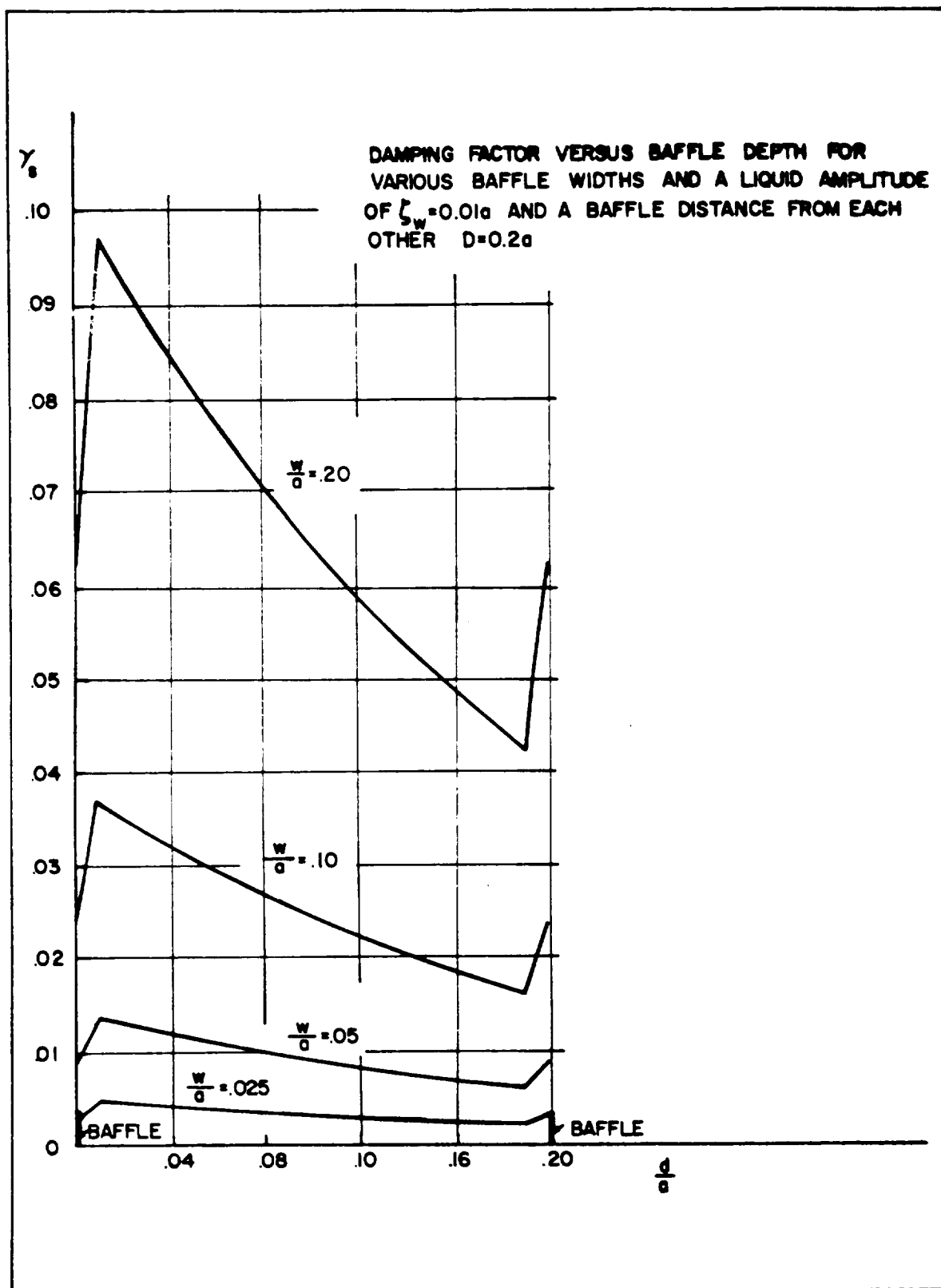


Figure 5-49

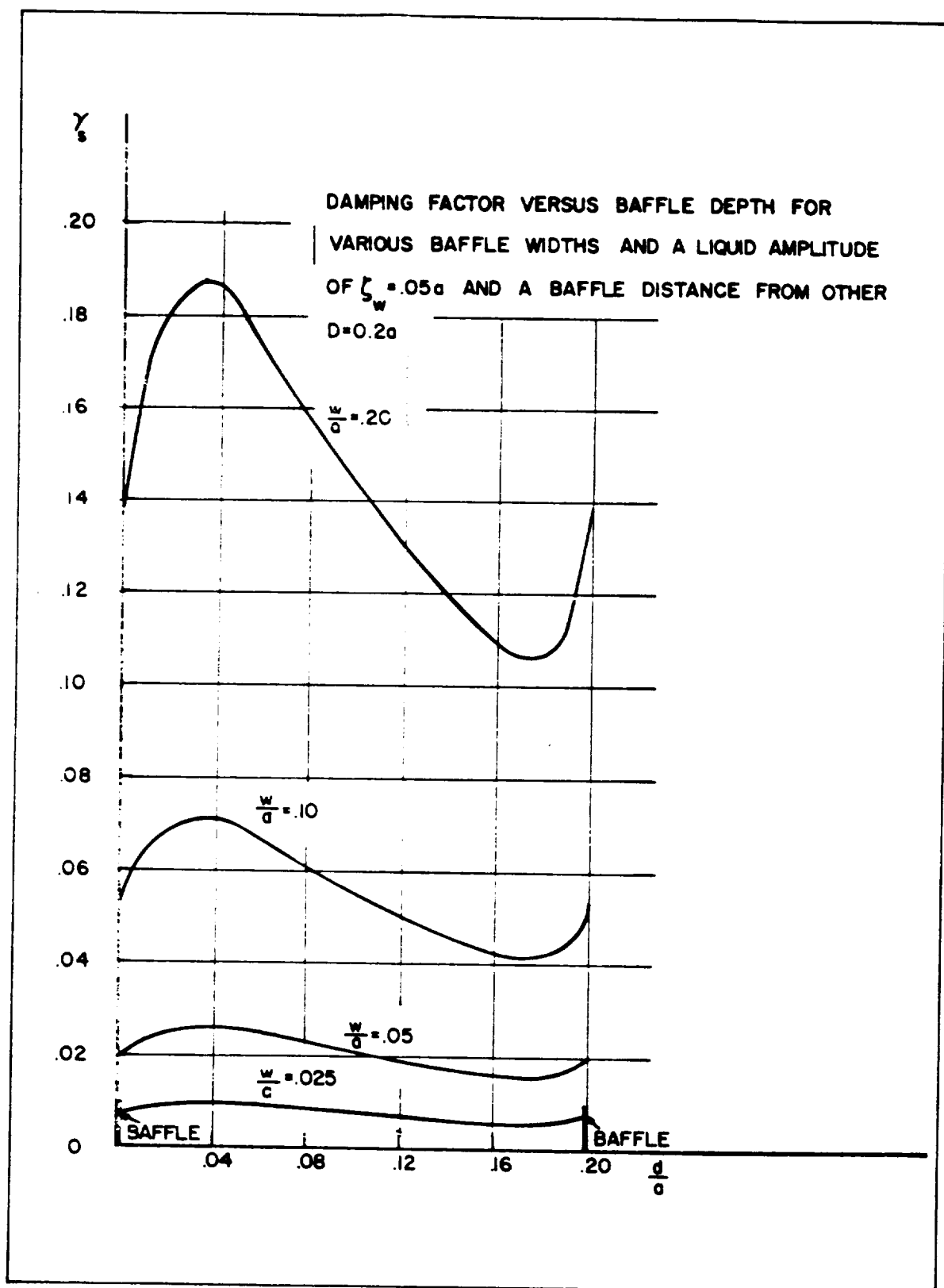


Figure 5-50

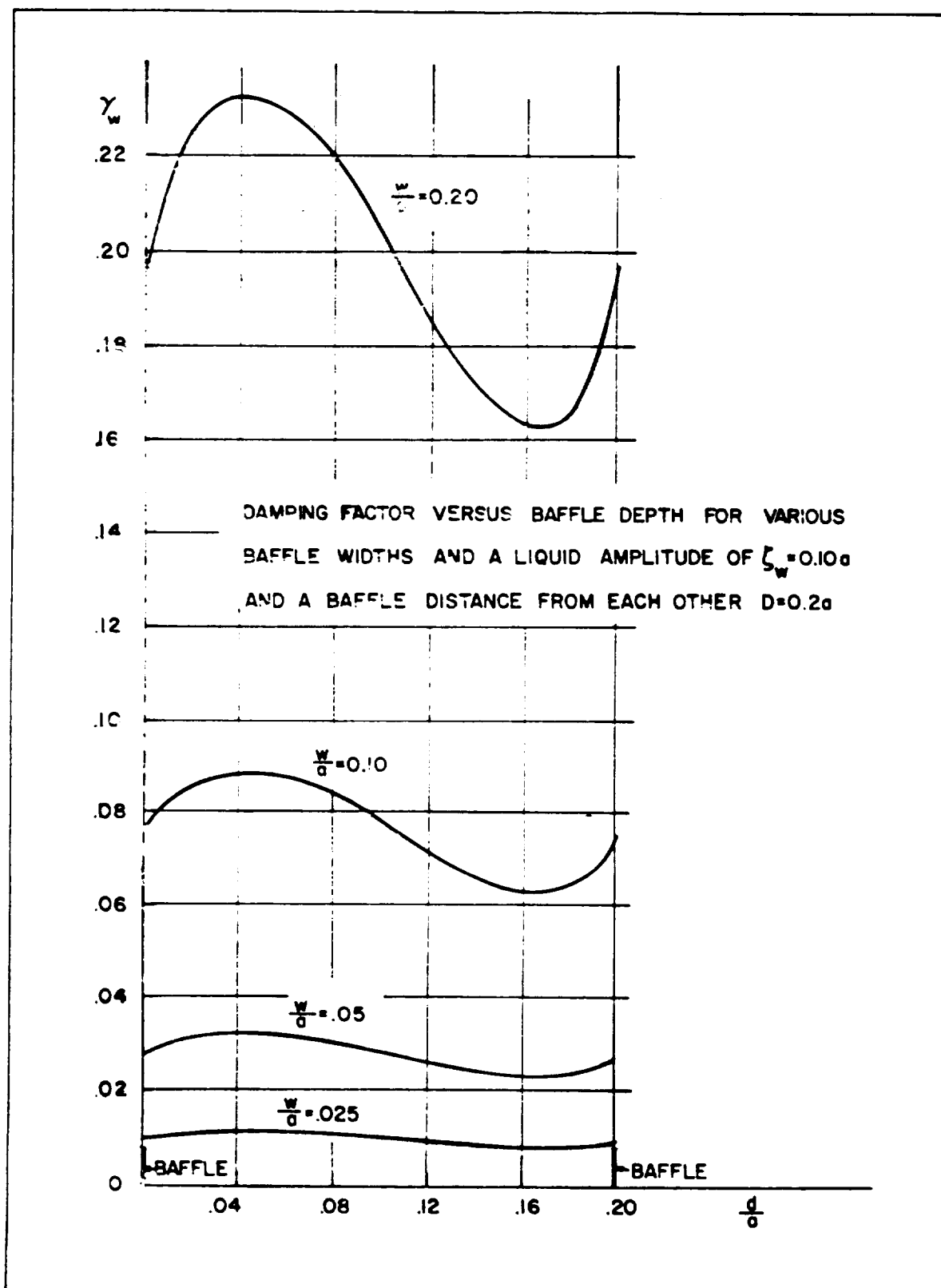


Figure 5-51

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR VARIOUS  
 BAFFLE WIDTHS AND A LIQUID AMPLITUDE OF  $\zeta_w = 0.2a$   
 AND A BAFFLE DISTANCE FROM EACH OTHER  $D = 0.2a$

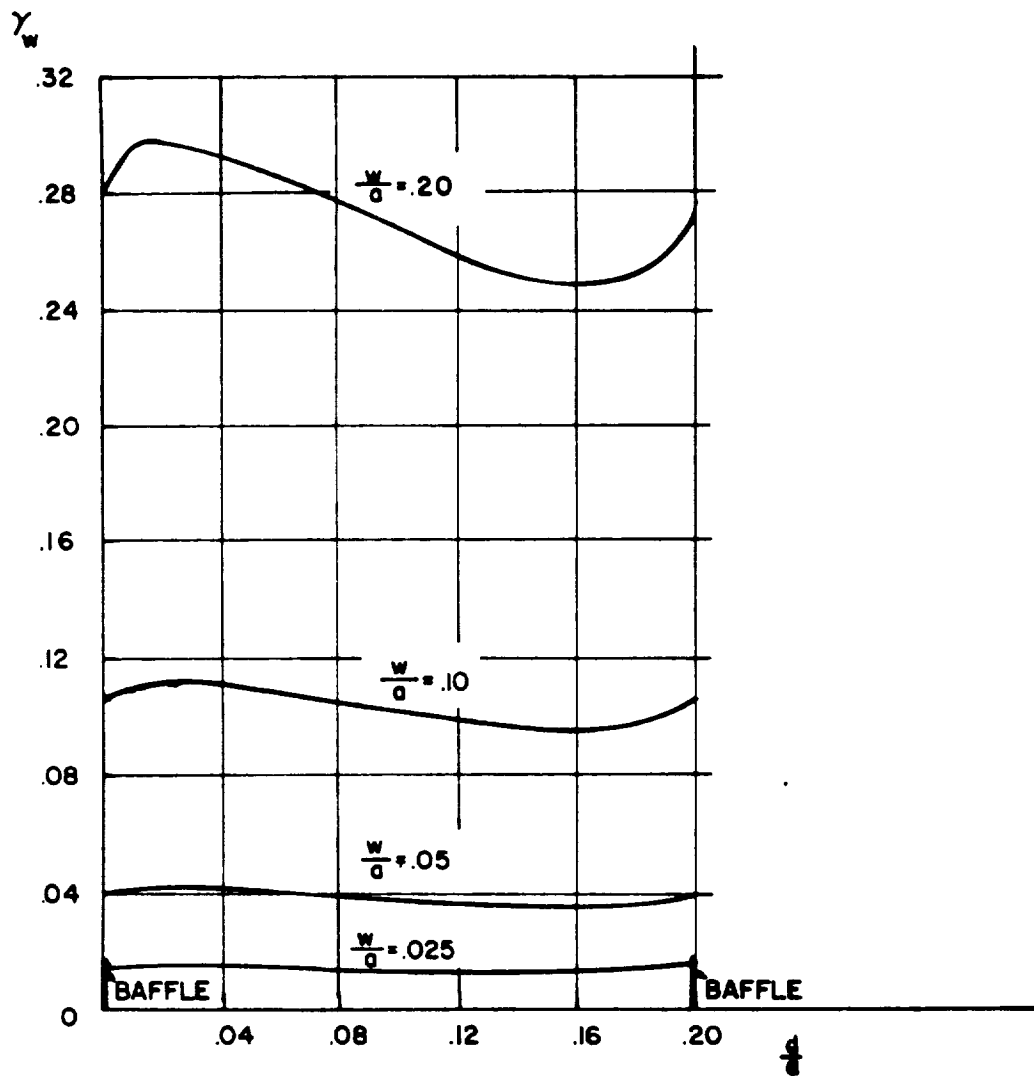


Figure 5-52

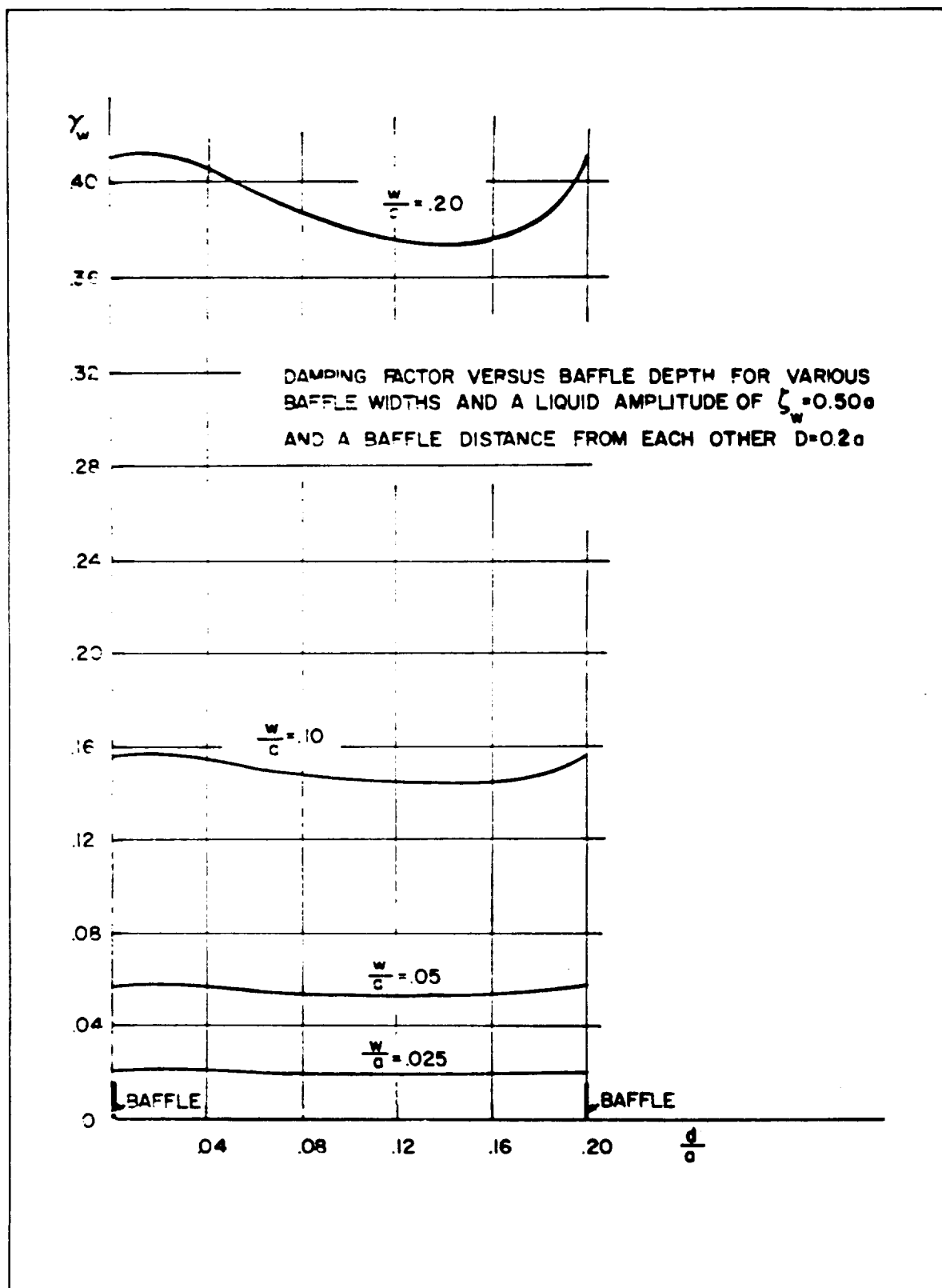


Figure 5-53

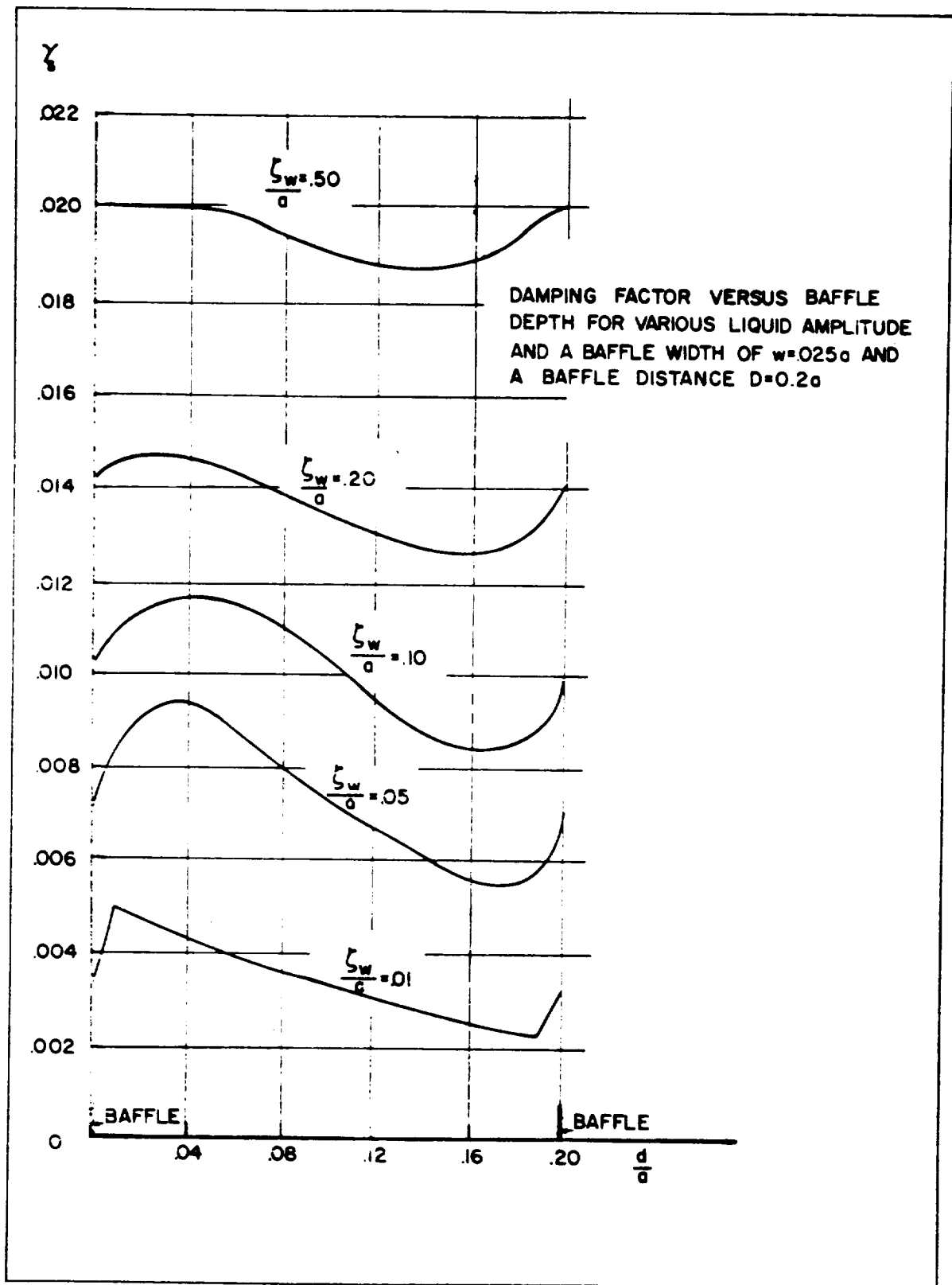


Figure 5-54

DAMPING FACTOR VERSUS BAFFLE DEPTH FOR VARIOUS  
LIQUID AMPLITUDE AND A BAFFLE WIDTH OF  $w=0.05a$   
AND A BAFFLE DISTANCE  $D=0.2a$

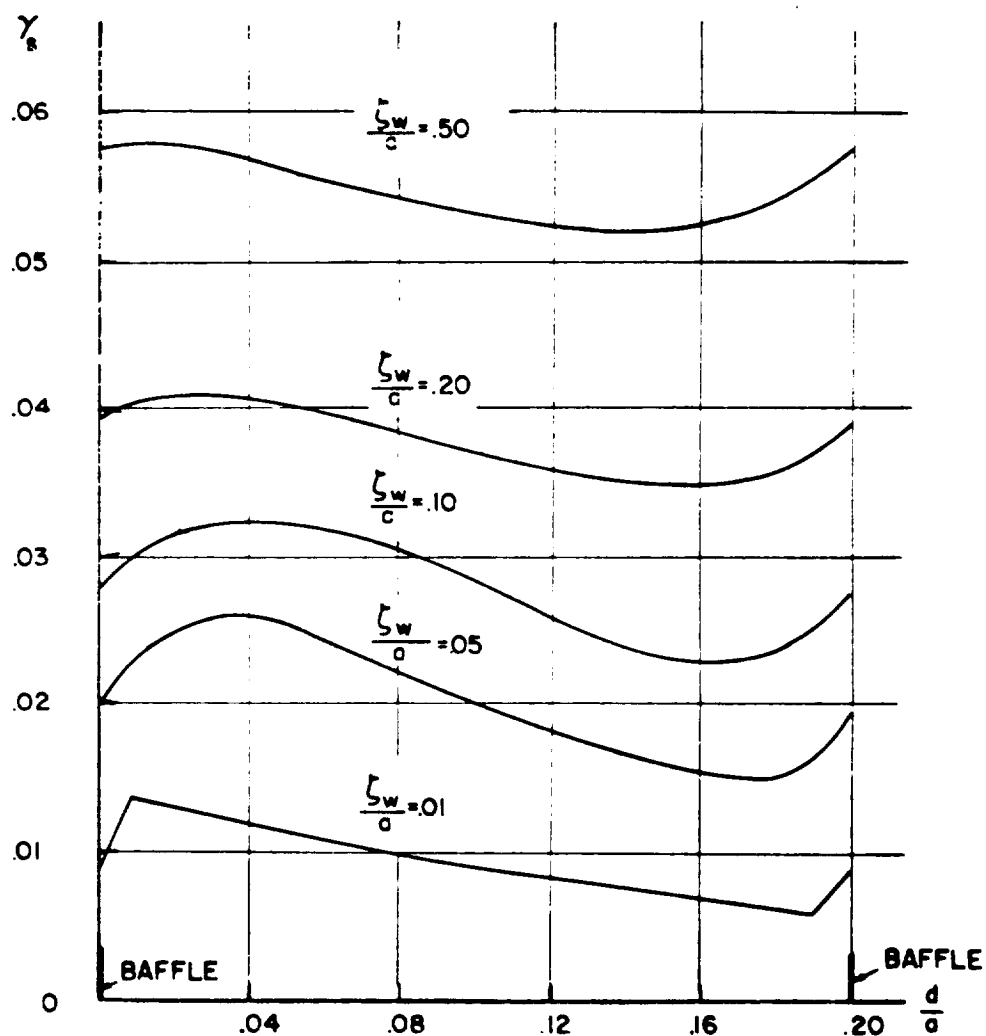


Figure 5-55

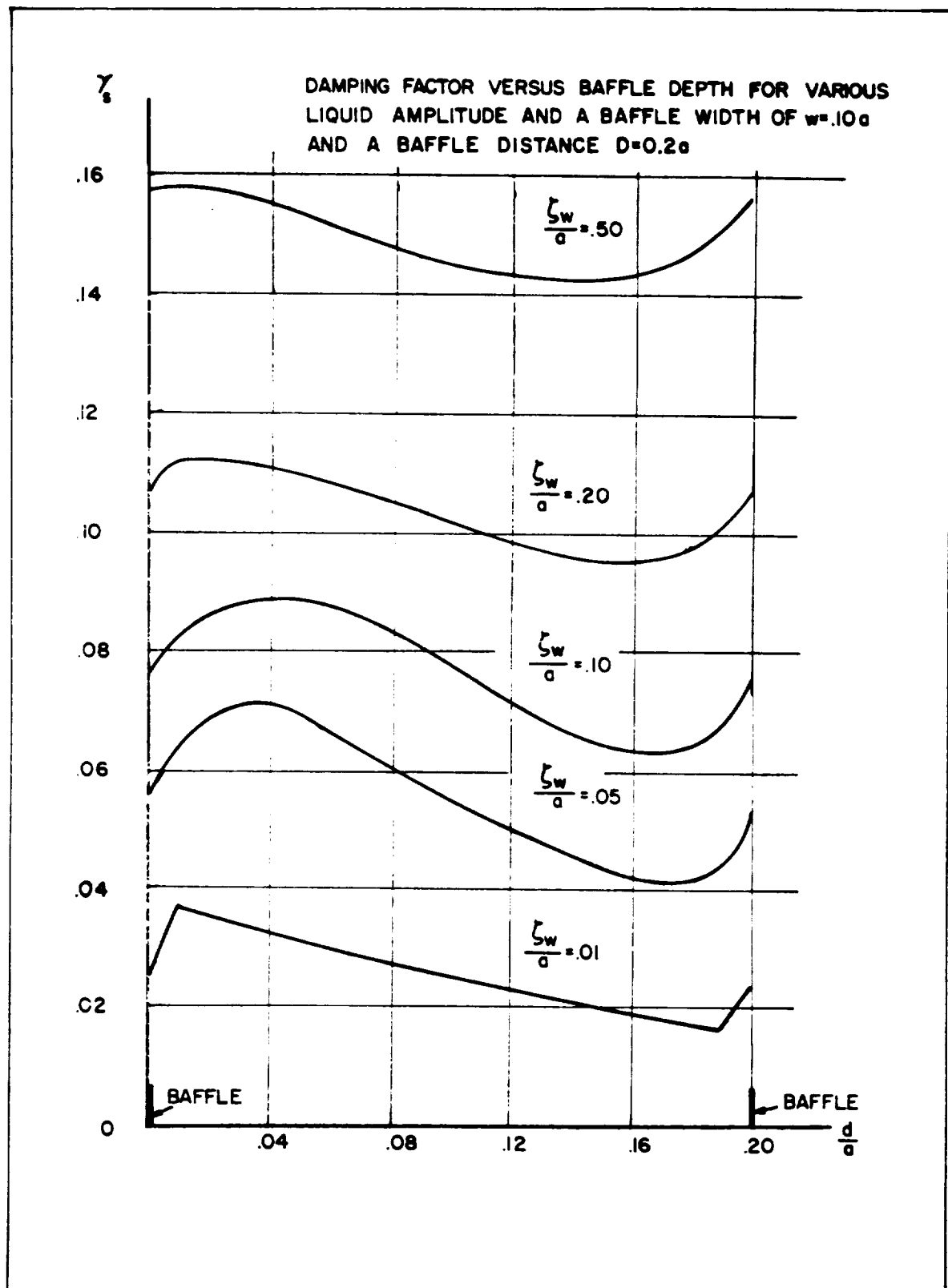


Figure 5-56



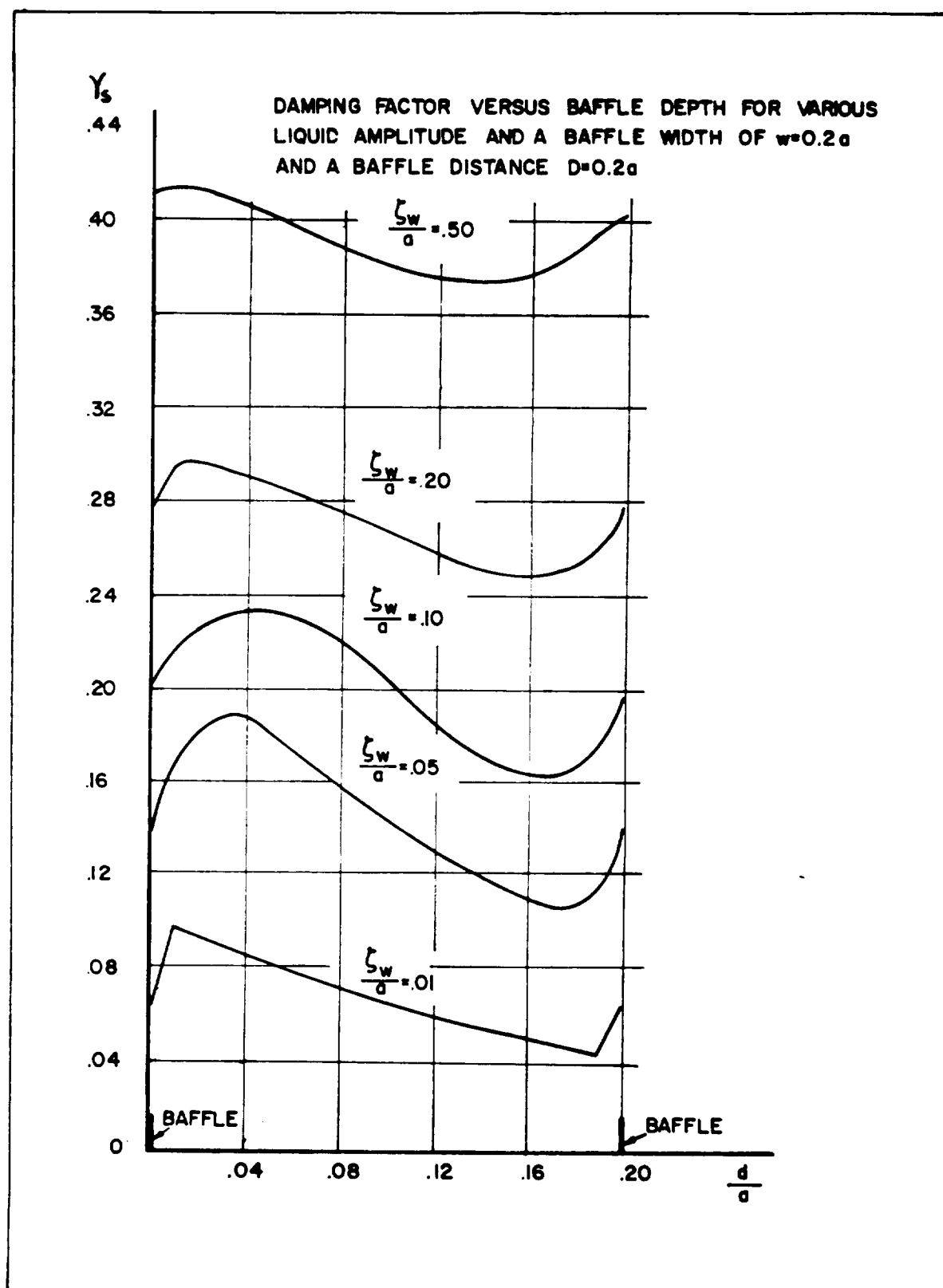


Figure 5-57

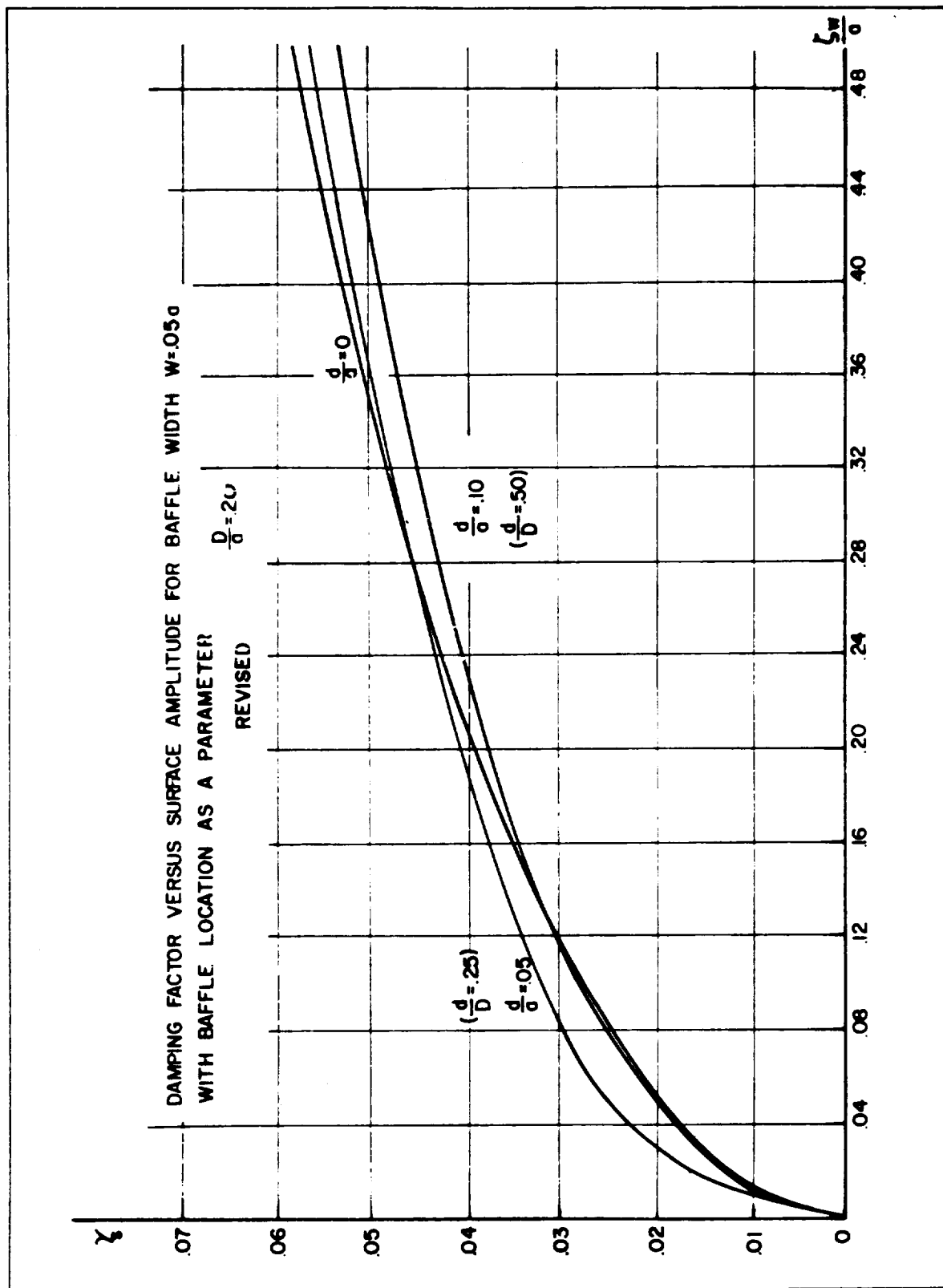


Figure 5-58

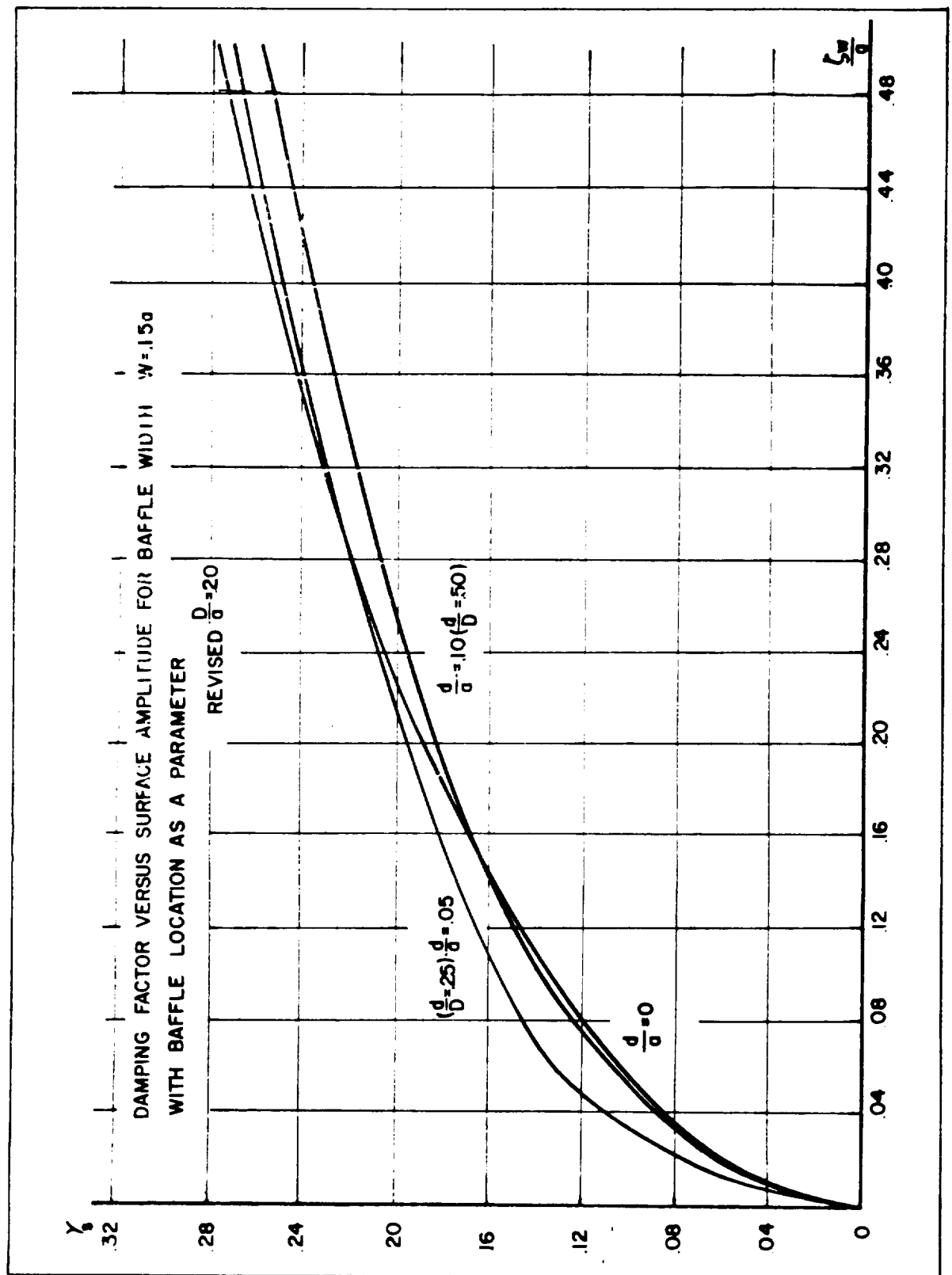


Figure 5-59

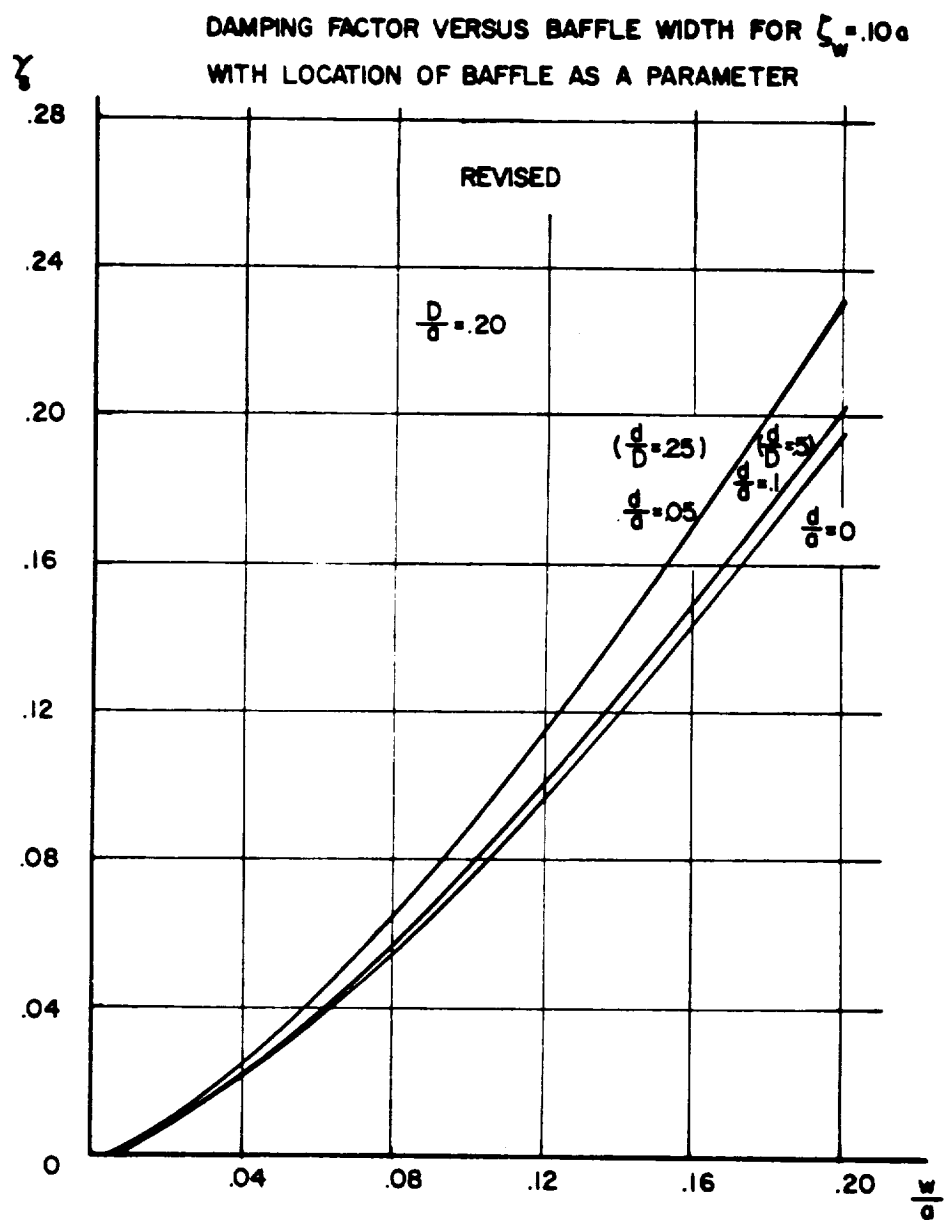


Figure 5-60

DAMPING FACTOR VERSUS BAFFLE WIDTH FOR  $\zeta_w = .30$   
WITH LOCATION OF BAFFLE AS A PARAMETER

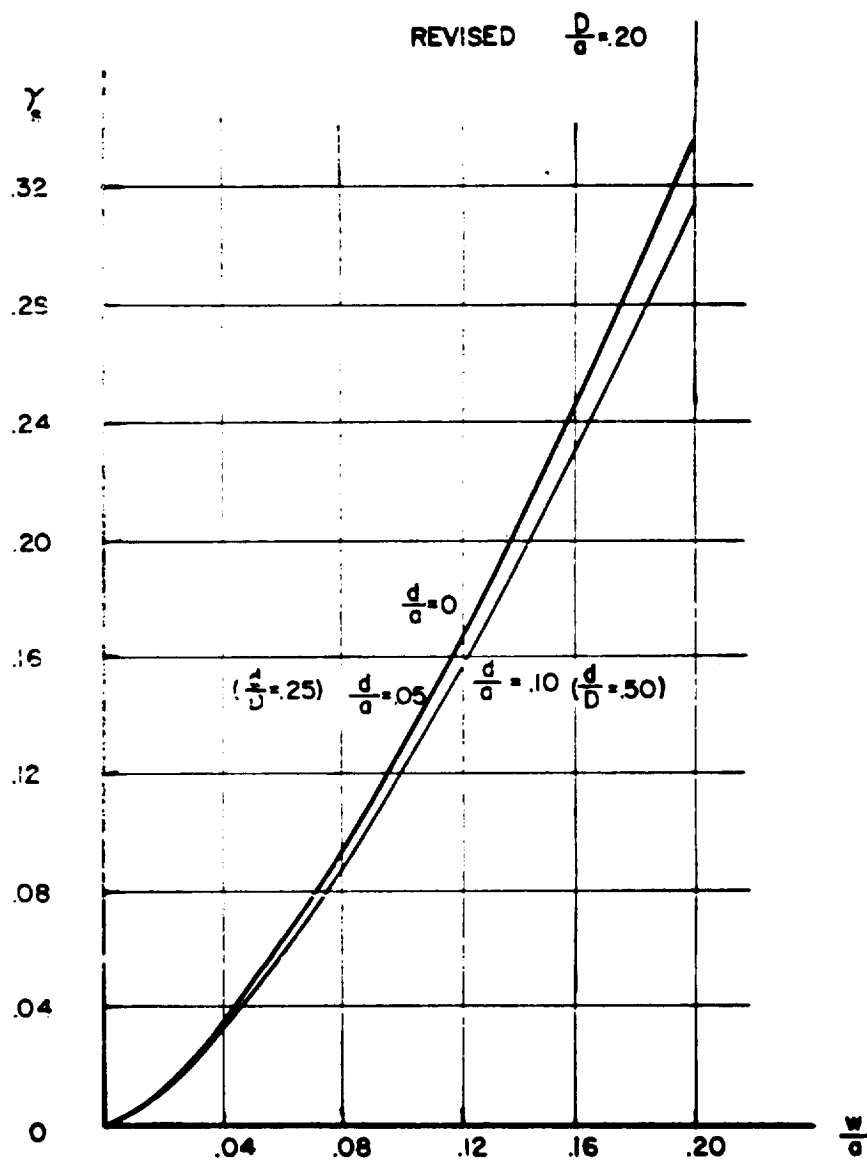


Figure 5-61

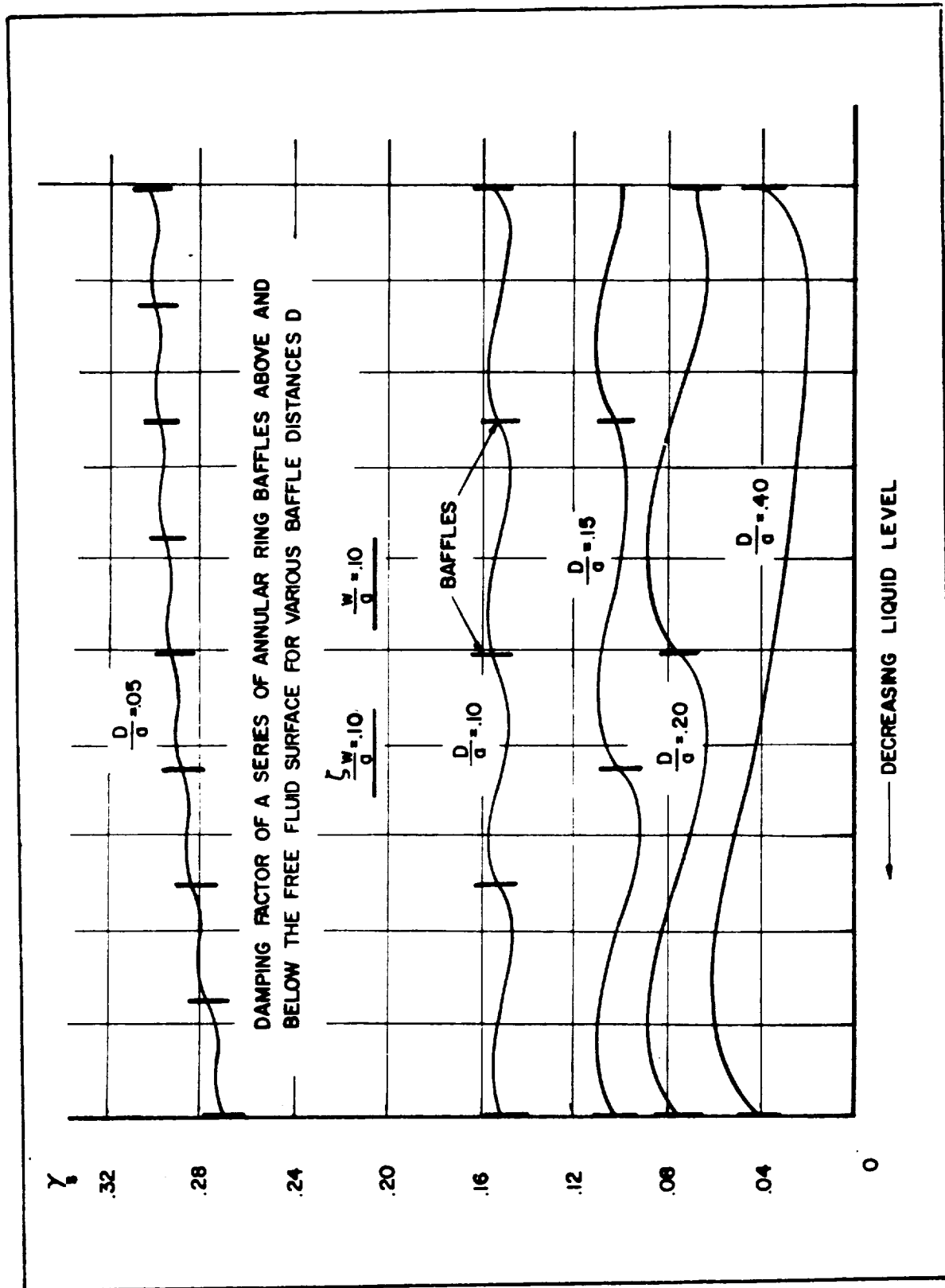


Figure 5-62

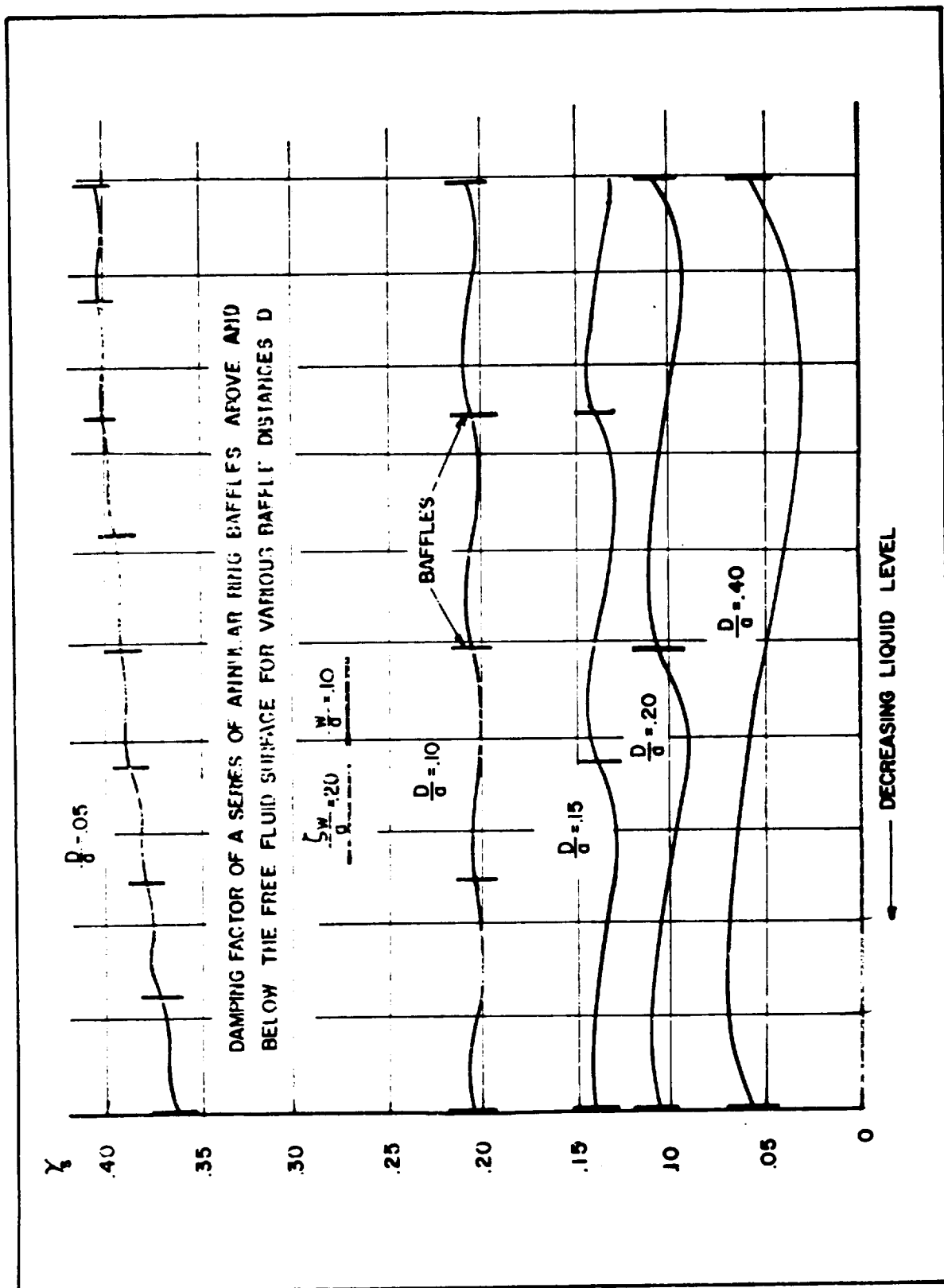
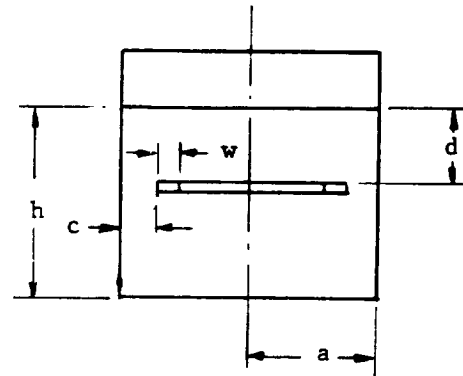
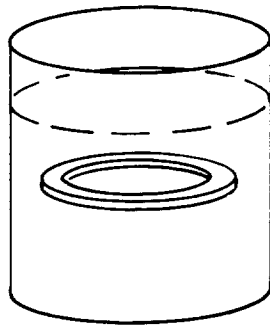


Figure 5-63

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.3 Ring with Radial Clearance

where  $h$  = depth of liquid  
 $d$  = depth of baffle  
 $a$  = radius of right circular cylinder  
 $w$  = ring width  
 $c$  = radial clearance

References: (35)

Investigations by Silveira, Stepes and Leonard (35) show that for a given baffle depth:

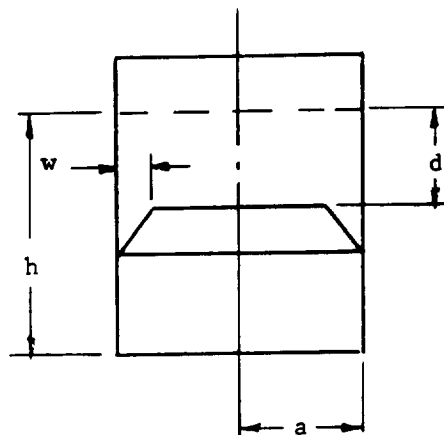
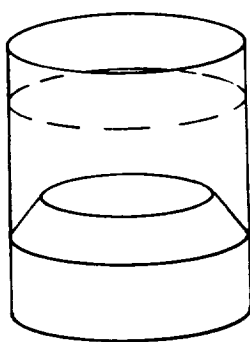
1. With the radial clearance constant:
  - (a) The damping factor increases as the baffle width increases (Figure 5-11)
  - (b) The frequency increases as the baffle width decreases (Figure 5-16).
2. With the baffle width constant:
  - (a) The damping factor decreases as the radial clearance increases (Figure 5-11)
  - (b) The frequency increases as the radial clearance decreases (Figure 5-16).



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.3.4 Conic Section



where  $h$  = depth of liquid

$d$  = depth of conic section below undisturbed free-surface

$a$  = radius of right circular cylinder

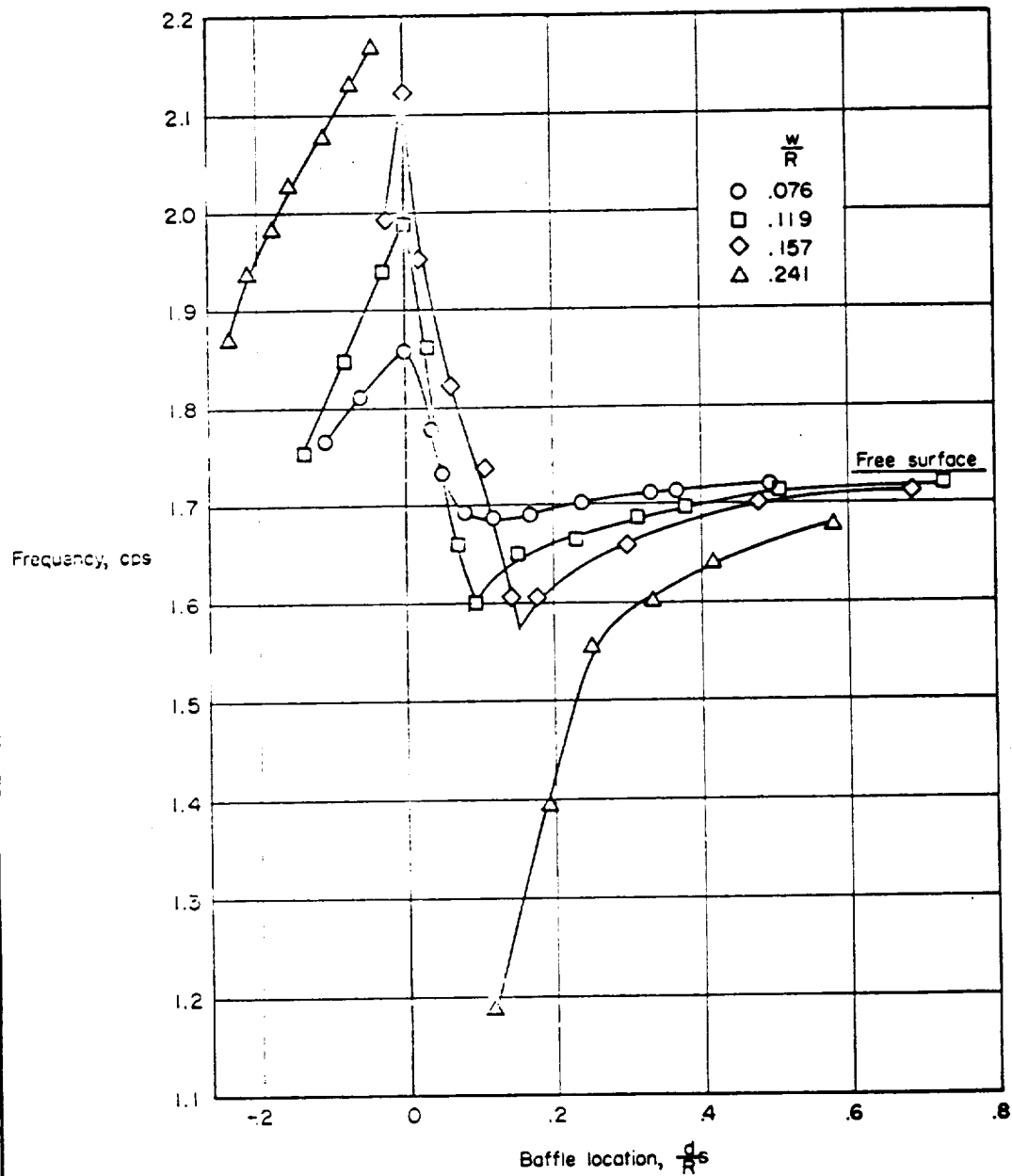
$w$  = conic section width

References: (35, 42, 43, and 83)

Investigations by Silveira, Stephens and Leonard (35) show that:

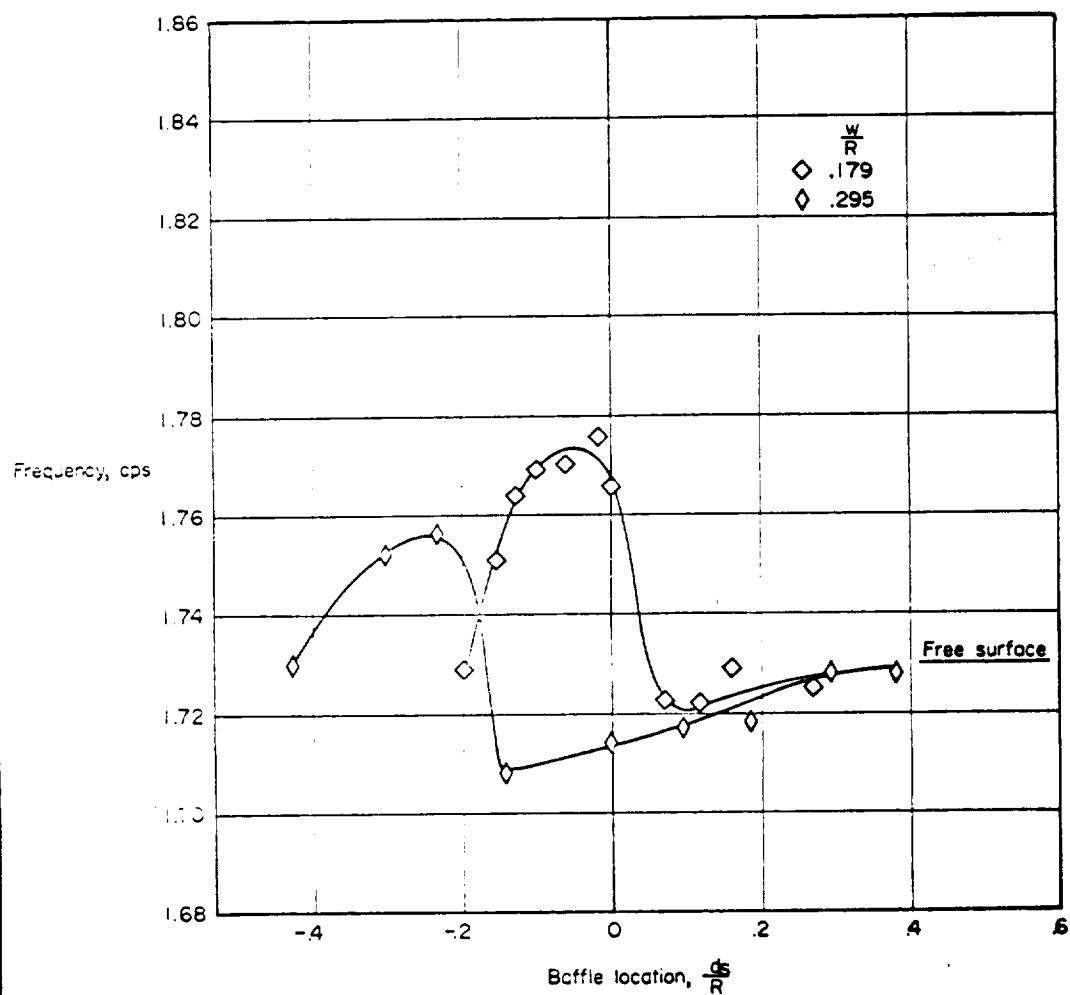
1. For a given baffle depth:
  - (a) The damping factor of a perforated and nonperforated conic section increases as the baffle width increases (Figure 5-68).
  - (b) The frequency for a perforated and nonperforated conic section increases as the baffle width decreases (Figure 5-64 and 5-67).
2. Although the damping provided by the conic sections is slightly higher than that for the ring baffles, the surface area of a conic section having the same baffle width as that of the ring baffle is considerably higher.

Investigations by Liu (83) give a conformal mapping solution for the pressure distribution on the conic section (article 5.1.7.3).



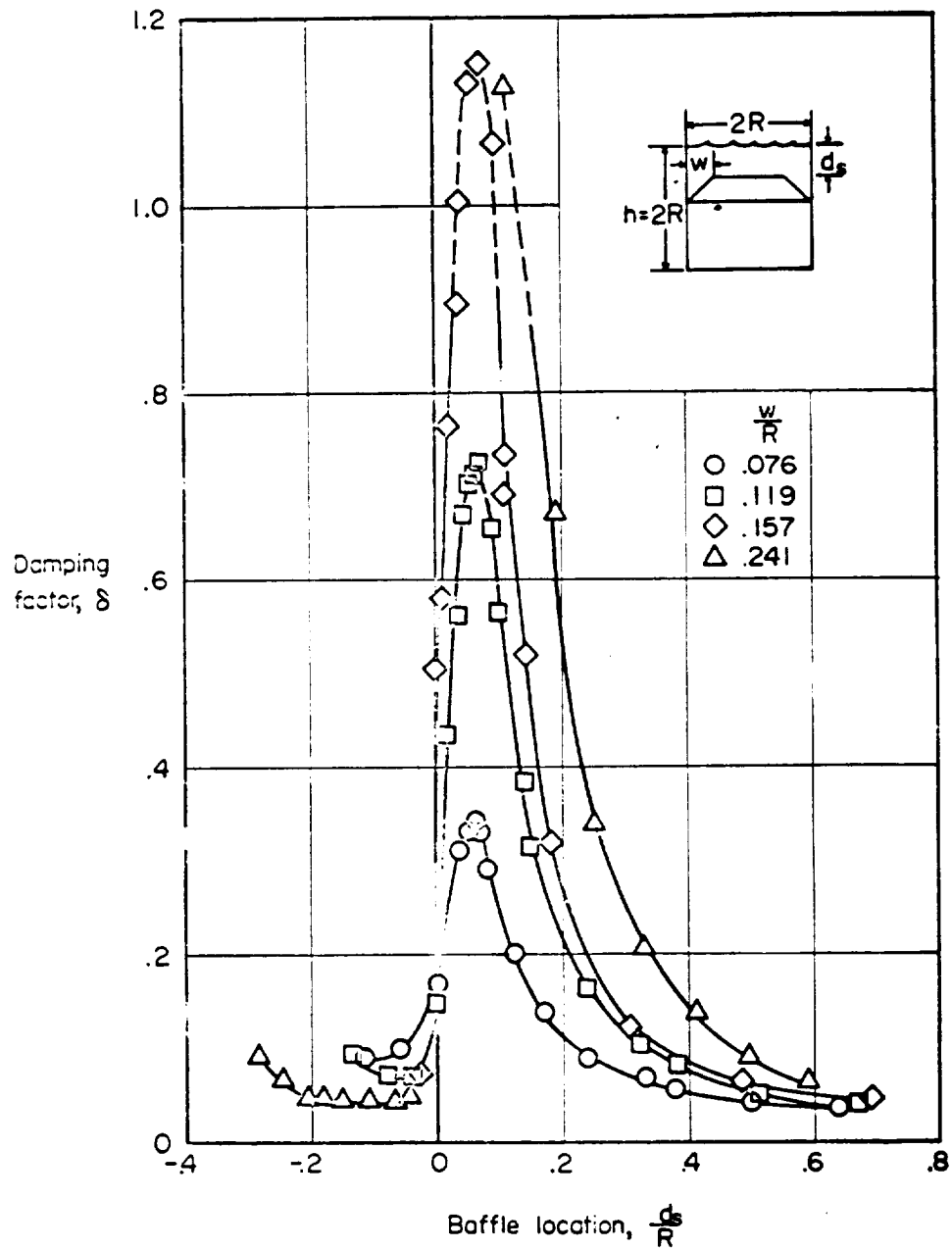
Variation of frequency with baffle location for conic-section baffle. (35)

Figure 5-64



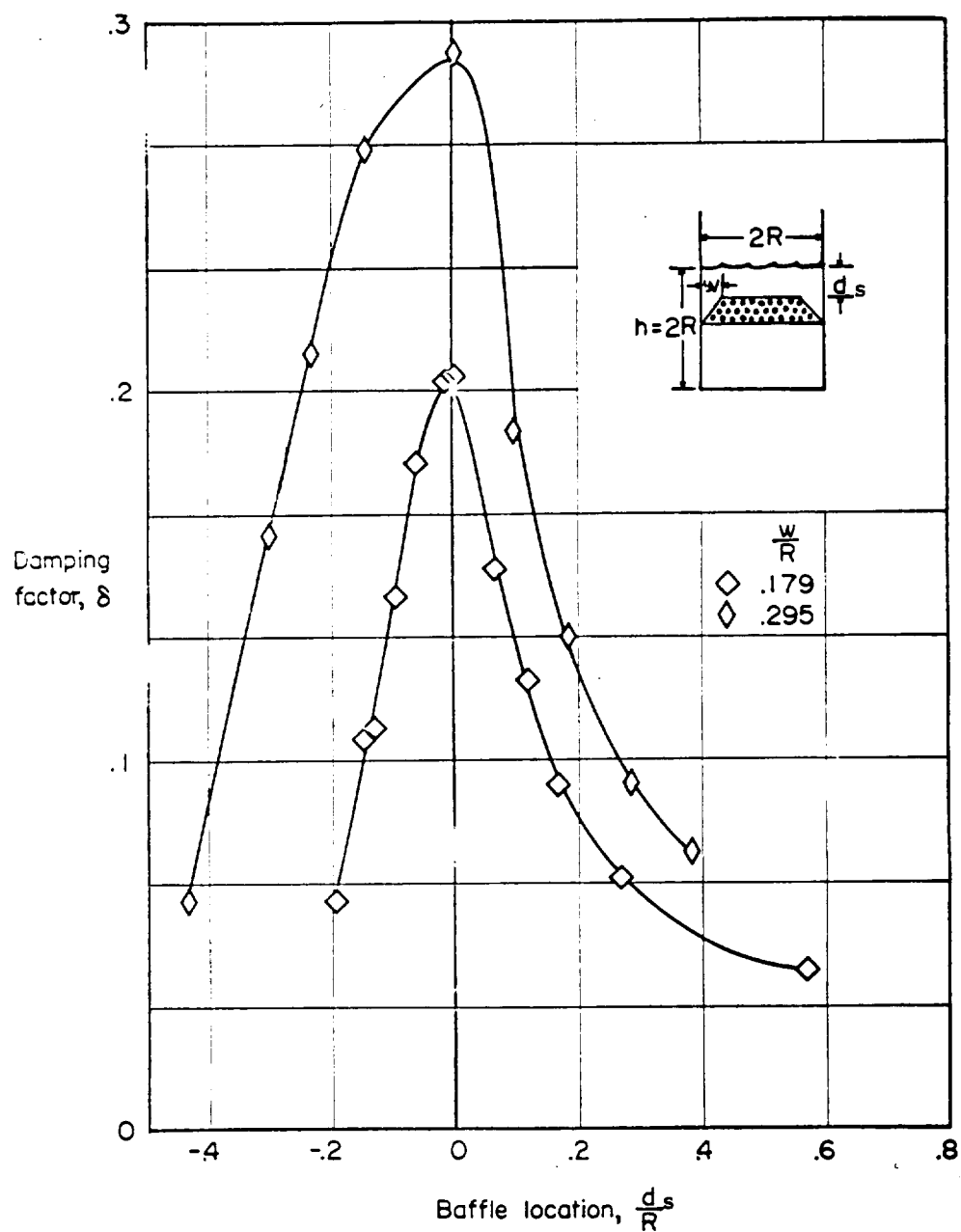
Variation of frequency with baffle location for perforated-conic-section baffle. (35)

Figure 5-65



Variation of damping factor with baffle location for conic-section baffle.  $R = 6$  inches;  $\frac{h}{R} = 2$ . (35)

Figure 5-66

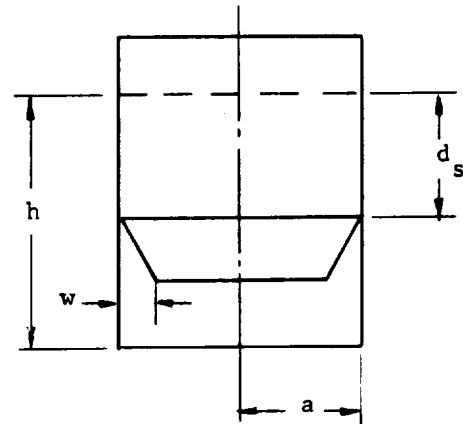
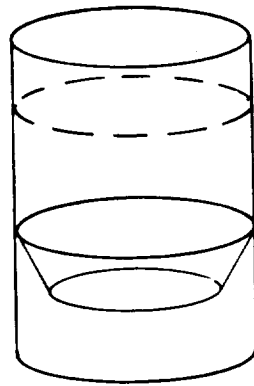


Variation of damping factor with baffle location for perforated-conic-section baffle.  $R = 6$  inches;  $\frac{h}{R} = 2$ . (35)

Figure 5-67

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.3.5 Inverted Conic Section

where  $h$  = depth of liquid

$a$  = radius of circular cylindrical tank

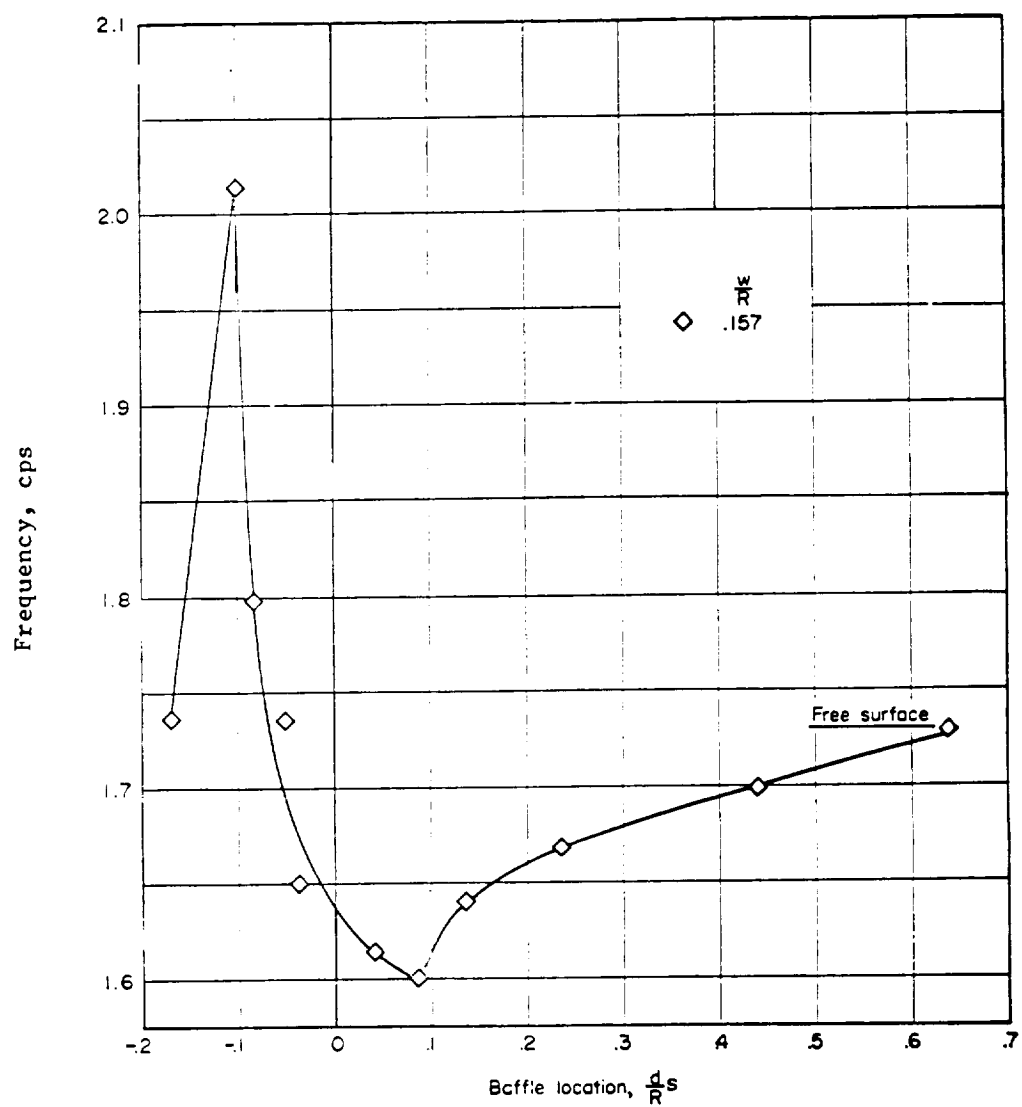
$d_s$  = depth of baffle

$w$  = inverted conic section width

References: (35, 42, and 83)

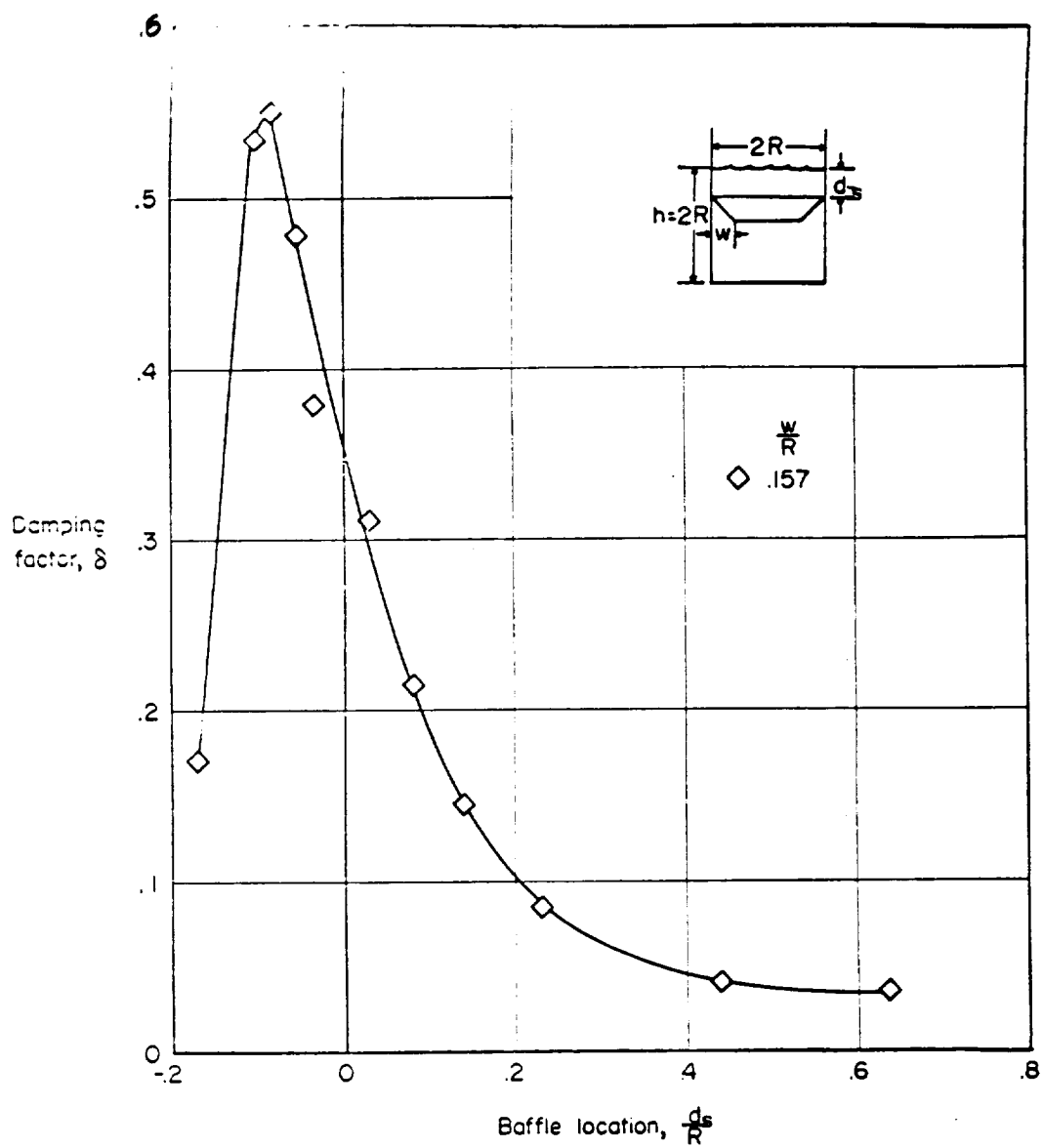
Investigations by Silveira, Stephens and Leonard (35) show the effect baffle width on both the frequency and the damping factor as a function of baffle depth (Figures 5.68 and 5.69).

Investigations by Liu (83) give a conformal mapping solution for the pressure distribution on the baffle.



Variation of frequency with baffle location for inverted-conic-section baffle. (35)

Figure 5-68



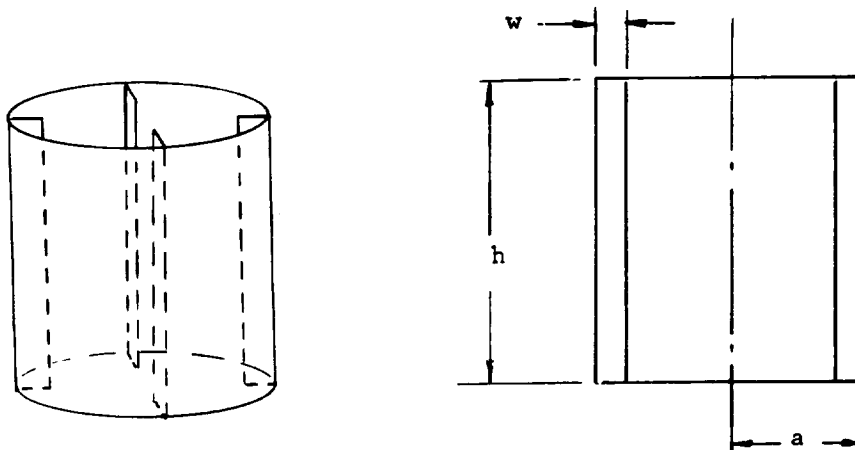
Variation of damping factor with baffle location for inverted-conic-section baffle.  $R = 6$  inches;  $\frac{h}{R} = 2$ . (35)

Figure 5-69



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.4 Cruciform Baffle

where  $h$  = depth of liquid

$a$  = radius of right circular cylinder

$w$  = cruciform width

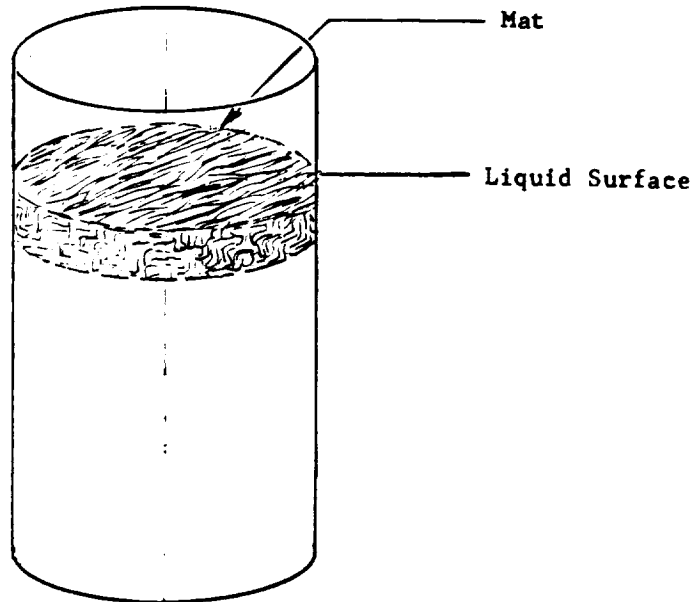
References: (35)

Cruciform baffles are most effective against rotary slosh which has large horizontal components of fluid velocity near the tank wall. Cruciform baffles are also slightly effective against lateral slosh since large horizontal fluid velocities exist where the node line of the first slosh mode intersects the tank walls.

Investigations by Silveira, Stephens and Leonard (35) indicate, as would be expected, that damping is independent of fluid depth except for shallow fluid depths.

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.5.1 Mat

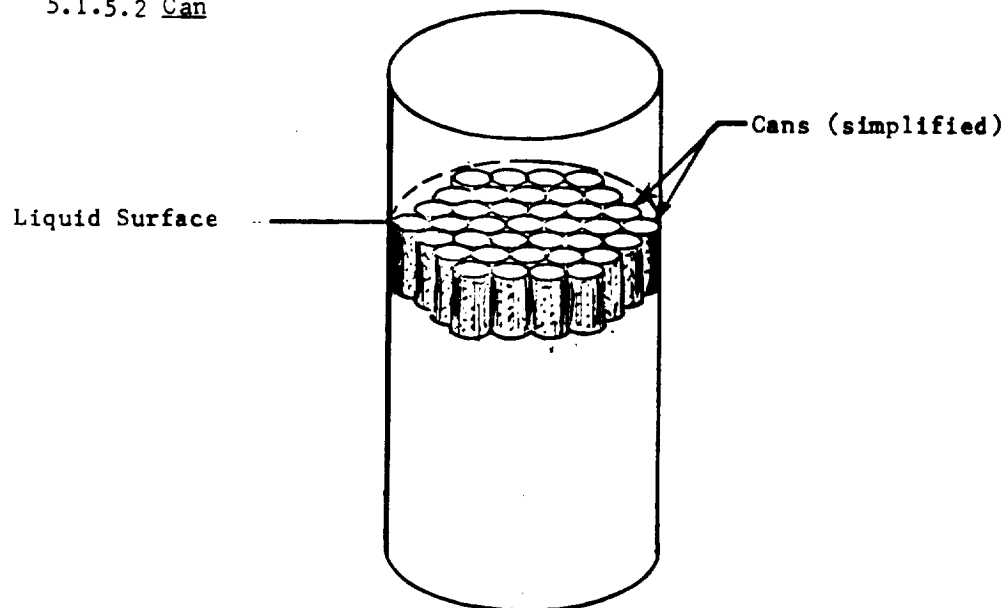
References: (40, 42, 43)

Investigations by Eulitz (40) have been made using commercial coco-fiber mats cut into a shape corresponding to the cross-sectional area of the tank. Several layers were used to attain the required thickness and hollow, airtight aluminum spheres were inserted between the layers to provide the necessary buoyancy. The device was highly effective in damping the liquid amplitude; however, because of the following, it is impractical for missile application. The inner walls of a propellant tank are usually obstructed by stiffener rings, pipelines and other equipment. Therefore, if a floating device is to be used, it must be capable of adapting to the changing cross-sectional area of the container.

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.5.2 Can



References: (40, 42, 43)

Investigations by Eulitz (40) have been made using cans made of perforated sheet metal and enclosing a hollow air-tight metal ball of slightly smaller diameter, which provides the necessary buoyancy. The optimum dimensions of a can necessary for adequate buoyancy and thereby the total number of cans is determined in the following manner. The fraction of submerged volume of the enclosed ball  $\lambda$  is given by

$$\lambda = \frac{(4)\sqrt{3}(\delta\gamma n)}{3D} [3 + (3 + n)P]$$

where  $\delta$  = the thickness of the can wall

$\gamma$  = the specific gravity of the can material

$n$  = the number of cans along a tank diameter

$D$  = the tank diameter

$P$  = the percent perforation

For a particular set of values  $\delta$ ,  $\gamma$  and  $P$ , a graph of  $\lambda$  vs.  $D$  can be plotted consisting of a family of curves corresponding to various values of  $n$  (for example, see Figure 5-73). The curve passing nearest to the intersection of the desired mean values of  $\lambda = 0.5$  and the tank diameter  $D$  under consideration will give the optimum value of  $n$ . The total number of cans is then determined by

$$N = \frac{1}{4} (3n^2 + 1)$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.5.2 Can (continued)

and the can diameter is given by

$$d_1 = \frac{D}{n}$$

The cans are made of perforated aluminum sheet enclosing a hollow air-tight aluminum ball, of slightly smaller diameter, which provides the necessary buoyancy (Figures 5-70 and 5-71). The size and number of cans used in the tests are indicated by  $n = 7$  or  $N = 37$  (see Figure 5-72).

The containers are rigid-wall circular cylindrical tanks of diameter 17.5 and 25 inches, respectively.

The optimum dimensions of the can necessary for adequate buoyancy, and thereby the total number of cans, is determined as follows: the desired value of  $\lambda$  the fraction of submerged volume of the aluminum ball, is 0.5. Figure 5-73 gives an equation for  $\lambda$  as a function of the thickness of the can wall  $\delta$ , the specific gravity of the can material  $\gamma$ , the percent perforation  $P$ , the tank diameter  $D$  and the number of cans along a tank diameter  $n$ . The curves shown in Figure 5-73 enable the designer to determine the values for  $n$  and  $N$  corresponding to  $\lambda \approx 0.5$  as a function of  $D$  for a set of values of  $\delta$ ,  $\gamma$  and  $P$ . Note that the notation used in Figures 5-71, 5-72, and 5-73 is applicable only to these figures.

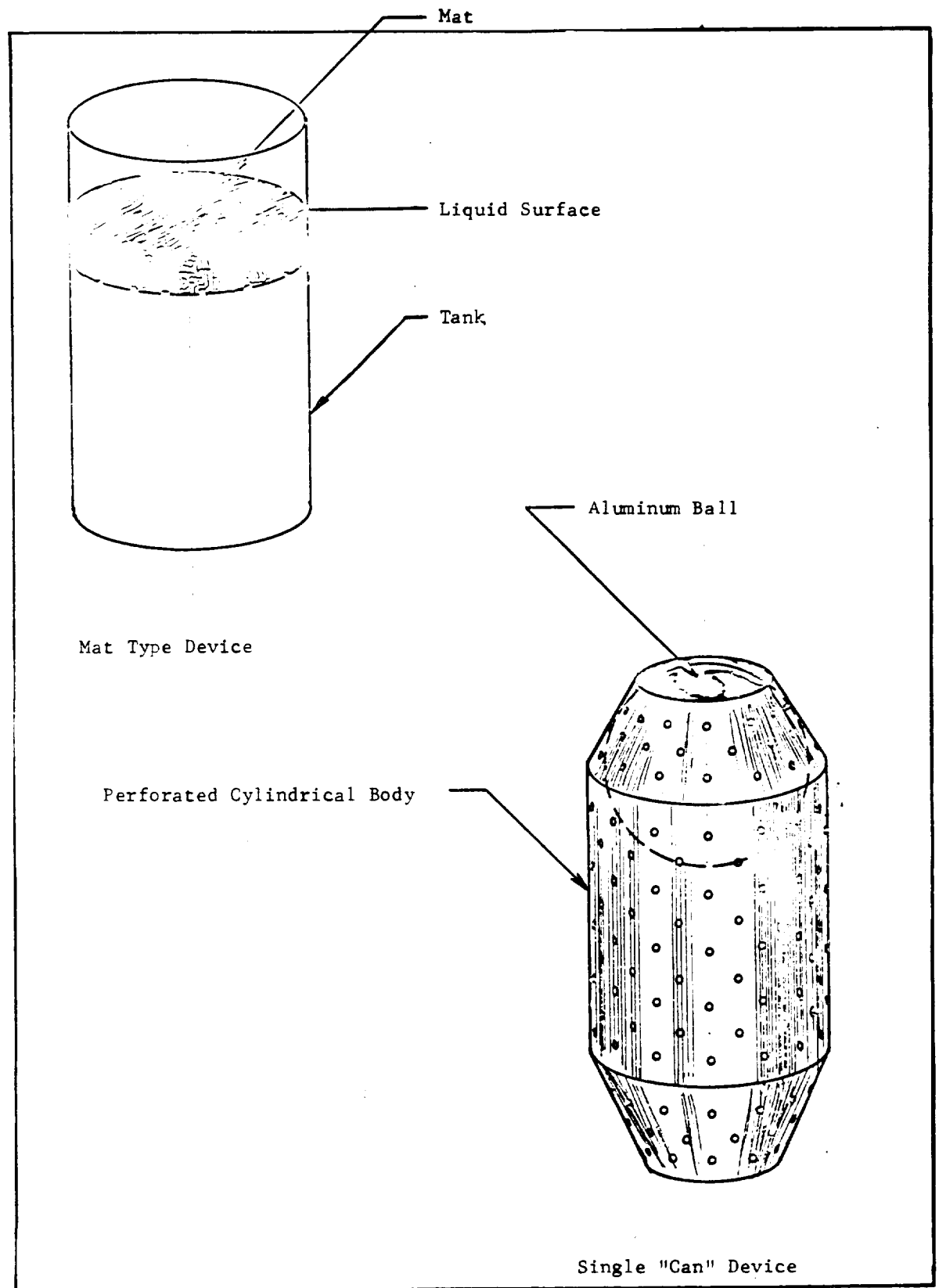


Figure 5-70 Mat type device and single can. (40)

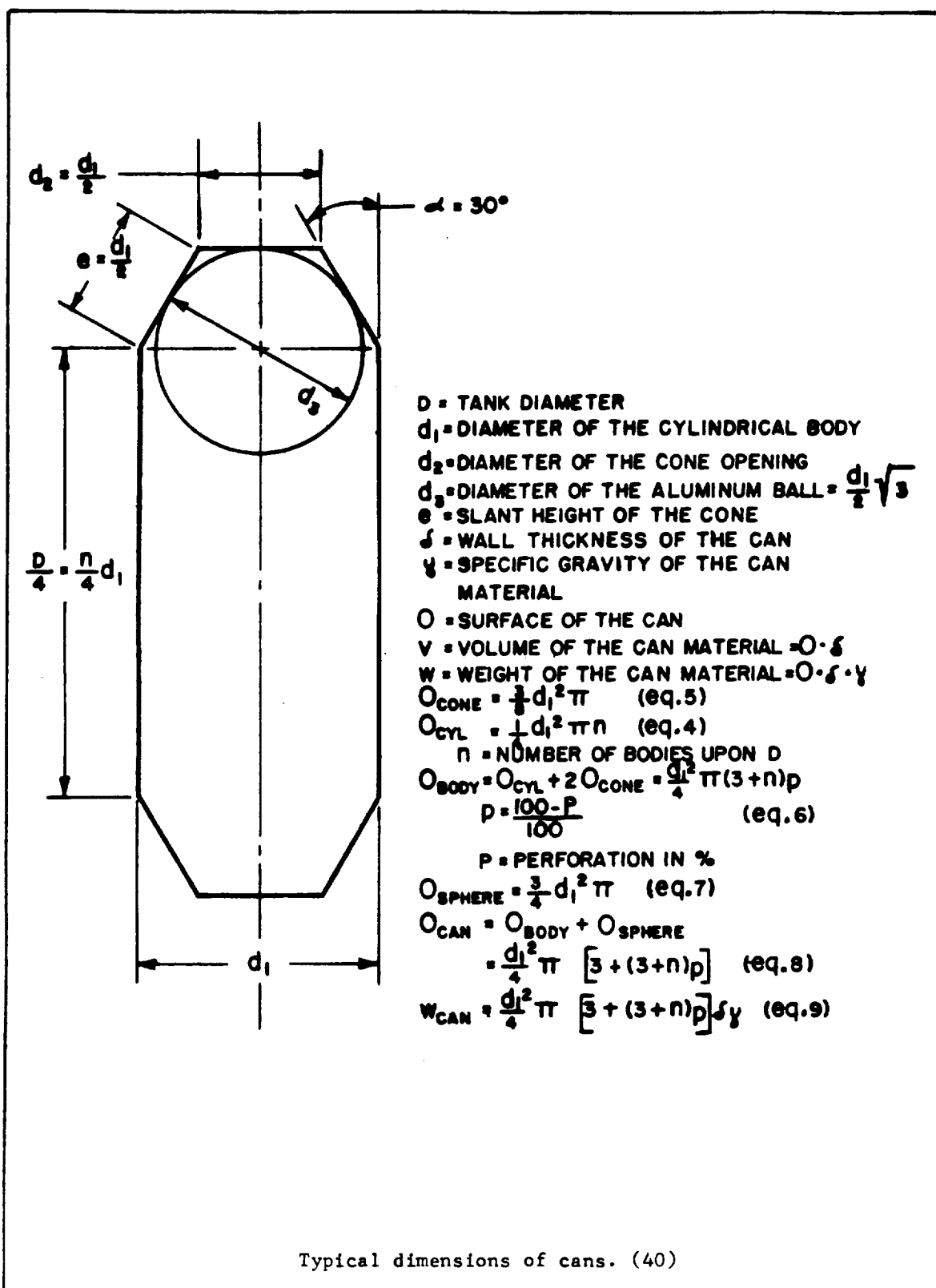


Figure 5-71

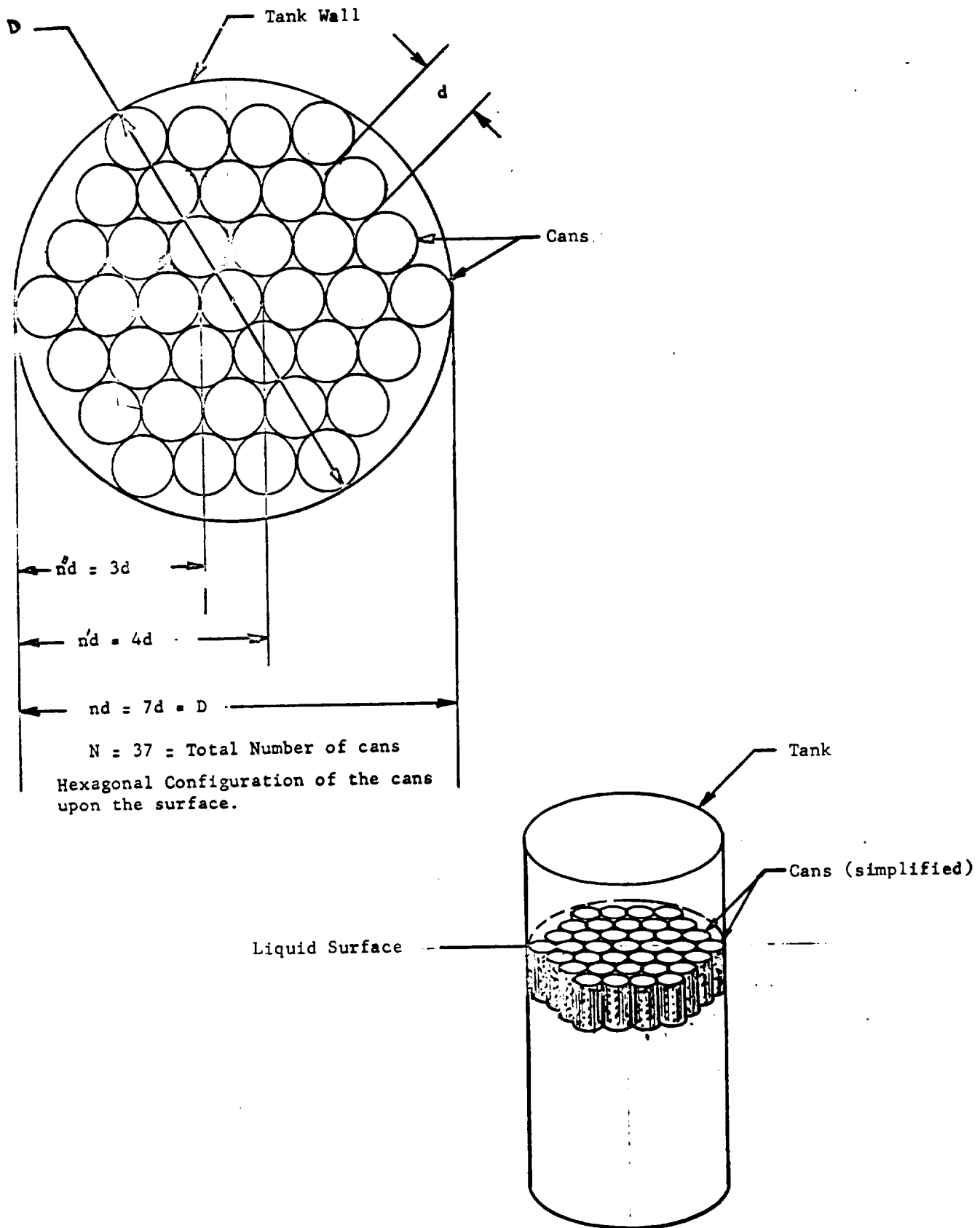


Figure 5-72 Distribution of cans within a tank. (40)

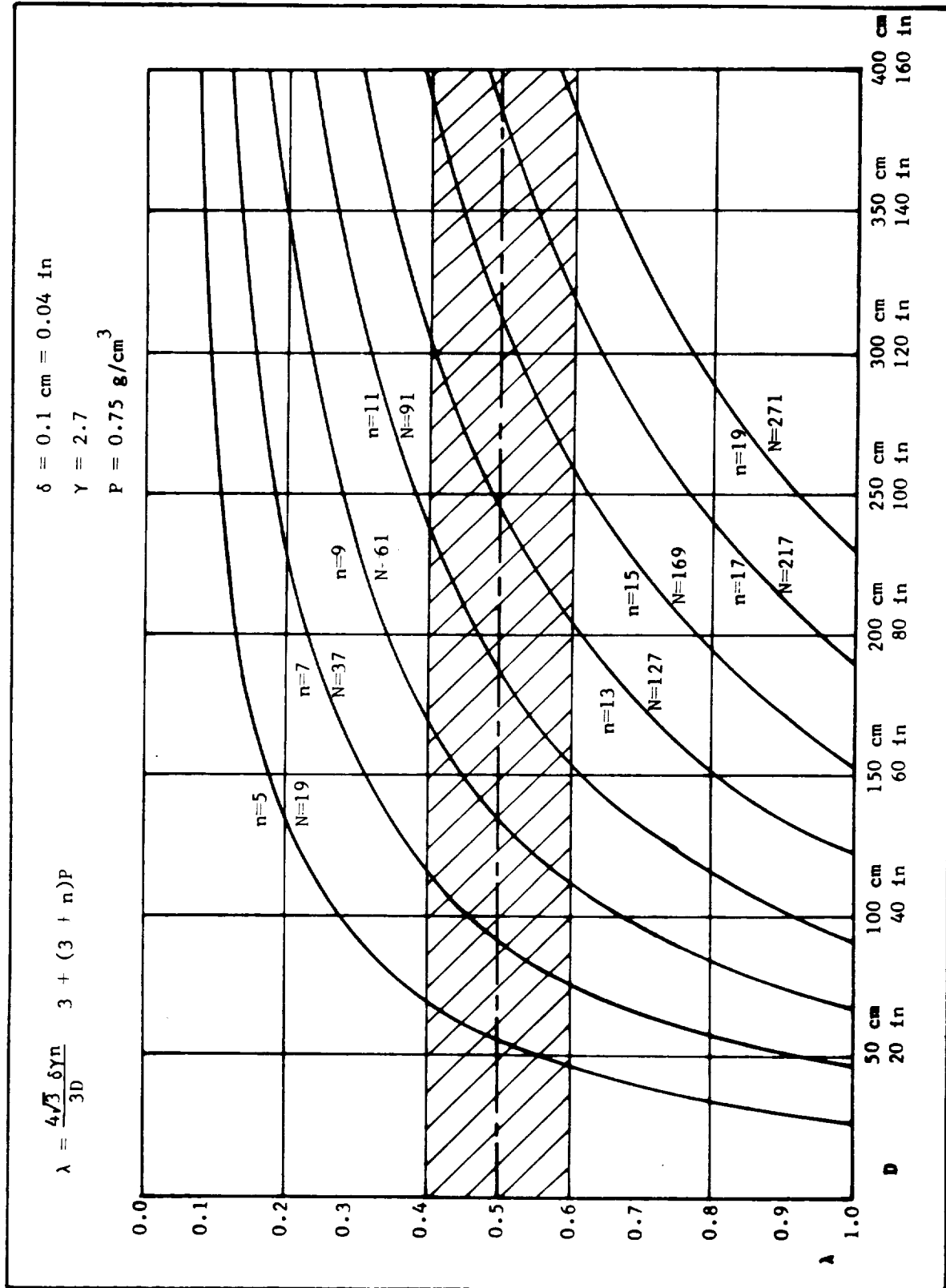


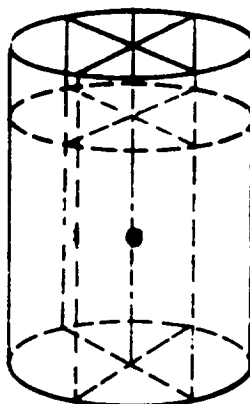
Figure 5-73

Optimum Number of Cans for Circular Cylindrical Tank. (40)



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

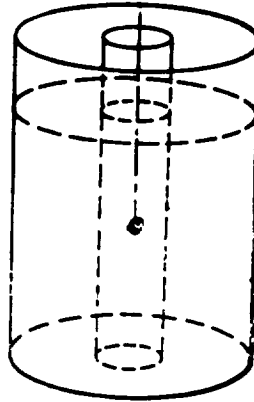
5.1.6.1 Cross PartitionsReferences: (4, 58)

Investigations by Bauer (4) indicate that cross partitions have a greater effect on the eigen frequencies than do concentric partitions. For the quarter-sectored tank, the oscillating propellant takes on various modes of vibration. Also, in the case of a quarter-sectored tank, the vibrating liquid mass is reduced to more than one-half of that of a circular cylindrical tank.

Investigations by Garza (58) indicate the effect of hole size on the frequency as a function of percent perforation (Figure 5-78 and 5-79) and the effect of both percent perforation and frequency on the damping ratio as a function of the excitation amplitude (Figure 5-80 and 5-81).

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.6.2 Concentric Partitions

References: (4, 52)

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.7 Pressure Distribution. The designer of a baffle system is interested in finding the pressure history of the fluid acting on the baffles because after the spacing and width of the baffles have been optimized, integration of the pressure will give the moments to be expected during flight. Thus, with the forces and moments acting on the baffle determined, the baffle material and its thickness can be chosen to withstand the load exerted on the baffles by the oscillating fluid, insuring, also, that the system pays a minimum weight penalty from the addition of the baffles.

5.1.7.1 Procedure I (28). The force per unit area on the ring is given by

$$\begin{aligned} p(\theta, t) &= C_D \cdot \frac{1}{2} \rho w^2 \\ &= C_D \frac{\rho}{2} \omega^2 f^2(-d) \cos^2 \theta \sin^2(\omega t) \end{aligned} \quad (5-9)$$

where  $w$  is the vertical component of the fluid velocity,  $f(z) = \sinh k(z+h)/\sinh(kh)$  and  $C_D$  is the local drag coefficient. Its value may be obtained from the empirical relation

$$C_D = 15 \sqrt{\frac{U_m T}{D}}, \quad 2 \leq \frac{U_m T}{D} \leq 20$$

and

$$C_D = 2, \quad \frac{U_m T}{D} \geq 100$$

where  $\frac{U_m T}{D}$  is the "period-parameter";  $U_m$  denotes the timewise maximum velocity,  $T$  the period,  $D$  the plate width, and

$$\frac{U_m T}{D} = \frac{[\omega \zeta_1 f(-d) \cos \theta] (2\pi/\omega)}{\alpha a} = \frac{2\pi f(-d)}{\alpha} \left( \frac{\zeta_1}{a} \right) \cos \theta$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

Substituting  $\omega^2 = kg \tanh(kh)$  into Eq. (5-9) and expressing the maximum pressure on the baffle as an equivalent head of the liquid, we have

$$Z_{\max} = \frac{P_{\max}}{\rho g} = \frac{1}{2} k a \tanh(kh) f^2(-d) (C_D)_{\max} \left( \frac{\zeta_1^2}{a} \right) \quad (5-10)$$

5.1.7.2 Procedure II (18). Analytical research has shown that the motion of the free surface of tanked liquid propellants is normal to the undisturbed liquid free surface. Thus, the maximum pressure and moments exerted by the liquid on the baffles occur when the mean free surface is at baffle level. That is, when the baffle depth, with respect to the free surface, is zero ( $d = 0$ ).

The dynamic pressure of the fluid acting at any point on a baffle in a cylindrical tank can be expressed as

$$p = \frac{2kg\zeta_1 \rho \cos \theta}{144 (\epsilon_1^2 - 1)} \left[ \frac{J_1(\epsilon_1 \frac{r}{a})}{J_1(\epsilon_1)} \right] e^{-\frac{\epsilon_1 d}{a}} \quad (5-11)$$

where

$p$  = dynamic pressure, psi

$kg$  = longitudinal acceleration,  $\text{ft/sec}^2$

$\rho$  = mass density of the liquid,  $\text{#m/ft}^3$

$a$  = radius of the cylinder, ft

$\phi_1$  = angle corresponding to sloshing amplitude

$\zeta_1 = a \phi_1$ , maximum amplitude of sloshing, ft

$\theta$  = angular coordinate

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.2 Procedure II (18) (continued).

It can be seen from Equation (5-11) that the maximum positive pressure and maximum negative pressure will occur at  $\theta = 0$ ,  $r=a$  and  $\theta = 180^\circ$ ,  $r=a$ , respectively. The analytical expression for the moments acting on the baffle is obtained from the dynamic pressure. It has the following form:

$$M_B(S) = \left[ \frac{\pi}{4} \rho (a^4 - a_o^4) S^2 X(S) - \sum_{n=1}^{\infty} \left( \frac{S^4 X(S)}{S^2 + \omega_n^2} \right) \left[ \frac{2\pi a^4 \cosh\left(\frac{\epsilon_n}{a}\right)(h-d)}{\epsilon_n^2 (\epsilon_n^2 - 1) \cosh\left(\frac{\epsilon_n h}{a}\right)} \right] \left[ K\left(\frac{a_o}{a}, \epsilon_n\right) \right] \right] \quad (5-12)$$

where

$$K\left(\frac{a_o}{a}, \epsilon_n\right) = \left\{ 1 - \left(\frac{a_o}{a}\right) \left[ \frac{J_1\left(\frac{\epsilon_n a_o}{a}\right)}{J_1(\epsilon_n)} \right] + \left(\frac{a_o}{a}\right)^2 (\epsilon_n) \left[ \frac{J_1'\left(\frac{\epsilon_n a_o}{a}\right)}{J_1(\epsilon_n)} \right] \right\} = \text{constant}$$

$S$  = Laplace transformation operation

$a_o$  = Inner radius of annular baffle, ft

For a circular cylindrical tank of arbitrary size it is convenient to express Equation (5-12) as a ratio of the baffle moment to the product of longitudinal acceleration  $kg$  and sloshing amplitude  $\zeta_1$ . That is,

$$\frac{|M_{B_1}|}{\rho a^3 k |\zeta_1|} = \left[ \frac{g e^{-\frac{\epsilon_1 d}{a}}}{4} \right] K\left(\frac{a_o}{a_1}, \epsilon_1\right) \quad (5-13)$$

where

$|M_{B_1}|$  = absolute value of the moment on the baffle corresponding to the first oscillation mode ( $n = 1$ )

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.2 Procedure II (18) (continued)

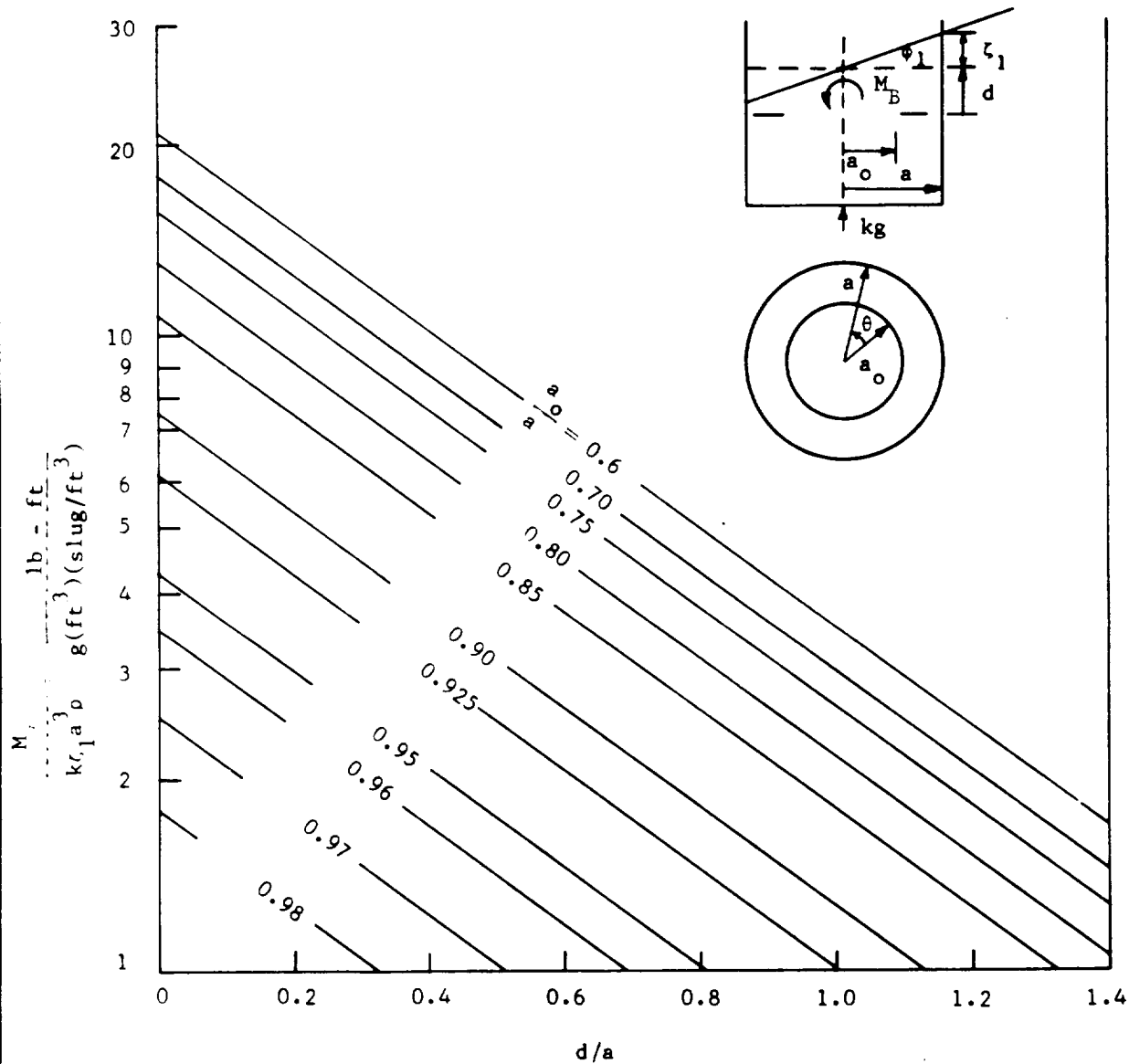
Graphical representation of Equation (5-13) can be found in Figure (5-74). It shows the baffle moment as a function of baffle width. The curves in this figure are valid only for fluid depth greater than one radius of the tank measured from the bottom. That is,  $\frac{h}{a} > 1$ . For  $\frac{h}{a} < 1$ , the following correction factor is recommended:

$$CF = \frac{e^{\frac{\epsilon_1 d}{a}}}{\cosh \frac{\epsilon_1 d}{a} + \sinh \frac{\epsilon_1 d}{a} \tanh \frac{\epsilon_1 d}{a}}$$

Then, the corrected moment expression becomes

$$M'_{B1} = CF \left\{ M_{B1} \left[ \text{Equation (5-13)} \right] \right\} \quad (5-14)$$

$$\frac{M}{k\zeta_1 a^3 \rho} = \frac{\pi}{4} g e^{-\frac{\epsilon_1 d}{a}} \left[ 1 - \frac{a_0}{a} \frac{J_1(\epsilon_1 \frac{a_0}{a})}{J_1(\epsilon_1)} + \left(\frac{a_0}{a}\right)^2 \epsilon_1 \frac{J_1'(\epsilon_1 \frac{a_0}{a})}{J_1'(\epsilon_1)} \right]$$



Moments Produced on Ring Baffles (18)

Figure 5-74

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.7.3 Procedure III (83). Conformal mapping is used to transform the geometrically complex representation of the baffle and the tank wall into a simple plane for which a solution to the Bernoulli equation for incompressible, nonviscous, unsteady flow is readily obtainable. Since this equation gives the velocity and pressure distribution on the baffle in the transformed plane it is then a simple matter to take the inverse transformation to find the solution in the original plane.

The transformation is obtained by using the Christoffel-Schwarz expression defined as

$$z = A \int_0^w (w-u_1)^{\alpha_1-1} (w-u_2)^{\alpha_2-1} \dots (w-u_k)^{\alpha_k-1} dw + B \quad (5-15)$$

where

$$z = F(w) = x + iy$$

$$w = u + iv$$

$x, y$  = Cartesian coordinates in  $z$ -plane

$u, v$  = Cartesian coordinates in  $w$ -plane

$A$  = Constant which determines the scale factor and rotation of the polygon

$B$  = Constant which determines the origin of the  $w$ -plane

$\pi\alpha_k$  = Interior angles of the polygon formed by the tank wall and the baffles in the  $z$ -plane

$u_k$  = Vertices of the polygon on the  $u$ -axis

The pressure distribution applicable to all three cases herewith given is as follows:



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

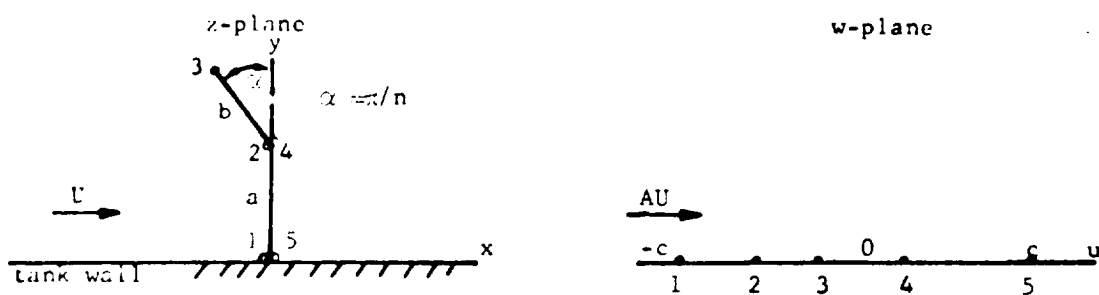
$$p(u,v) = \rho \left\{ |U| \omega \left[ \operatorname{Re} F(w) - Au \right] - |U|^2 \left( \frac{3}{2} - 2ARf + \frac{A^2 R}{2K} \right) \right\} \quad (5-16)$$

where  $\omega$  is the circular frequency,  $\rho$  the density of the fluid,  $U$  the velocity of the fluid acting on the baffle and  $f$ ,  $K$ , and  $R$  are parameters depending on the geometry of the system.

Baffle Type I. The baffle-tank wall system in the  $z$ -plane is mapped onto the  $w$ -plane by the transformation

$$z = A \int_0^w (w+c)^{-1/2} (w-u_2)^{-1/n} (w-u_3)(w-u_4)^{1/n} (w-c)^{-1/2} dw + B \quad (5-17)$$

The result is shown in the figure for Baffle Type I below.



Baffle Type I

## V. Propellant SLOSH Suppression

## 5.1 Anti-SLOSH Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

The parameters and constants for the pressure distribution given in Equation (5-16) are as follows:

$$K(u,o) = \frac{\sqrt{2A}}{2(c^2 - u^2)} \left| \frac{u-u_4}{u-u_2} \right|^{1/n}$$

$$R(u,o) = \frac{\sqrt{2}}{2A(u-u_3)^2} \left| \frac{u-u_2}{u-u_4} \right|^{1/n}$$

$$f(u,o) = \beta \sin(\theta_3/n)$$

$$g(u,o) = -\beta \cos(\theta_3/n)$$

$$A = b/I_2 \quad B = b_1 + ib_2$$

$$\theta_3(u,o) = \begin{cases} 0 & \text{for } -c \leq u \leq u_2, u_4 \leq u \leq c \\ \pi & \text{for } u_2 \leq u \leq u_4 \end{cases}$$

$$\beta(u,o) = \sqrt{2(c^2 - u^2)} (u-u_3)$$

$$I_o = \int_0^{u_3} |F'(w)| dw$$

$$I_2 = \int_{u_2}^{u_3} (w+c)^{1/2} (w-u_2)^{-1/n} (u_3-w)(u_4-w)^{1/n} (c-w)^{-1/2} dw$$

$$b_1 = -b(1+I_o/I_2) \sin \alpha; b_2 = b(1+I_o/I_2) \cos \alpha + a$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

The constants  $u_2, u_3, u_4$  can be determined from the integral expression for  $I_2$  above and from the following definite integrals:

$$\begin{aligned}
 I_1 &= \int_{-c}^{u_2} (w+c)^{-1/2} (u_2-w)^{-1/n} (u_3-w)(u_4-w)^{1/n} (c-w)^{-1/2} dw \\
 I_3 &= \int_{u_3}^{u_4} (w+c)^{-1/2} (w-u_2)^{-1/n} (w-u_3)(u_4-w)^{1/n} (c-w)^{-1/2} dw \\
 I_4 &= \int_{u_4}^{u_5} (w+c)^{-1/2} (w-u_2)^{-1/n} (w-u_3)(w-u_4)^{1/n} (c-w)^{-1/2} dw
 \end{aligned} \tag{5-18}$$

where  $I_1, I_2, I_3$ , and  $I_4$  are definite integrals which are proportional to the length of the sides of the polygon ABCDE corresponding to polygon 12345 in the transformation for Baffle Type I. Thus,

$$\frac{\overline{BC}}{\overline{AB}} = \frac{I_2}{I_1} = \frac{b}{a} = \lambda_2 \quad \frac{\overline{CD}}{\overline{AB}} = \frac{I_3}{I_1} = \frac{b}{a} = \lambda_3 \quad \frac{\overline{DE}}{\overline{AB}} = 1 = \lambda_4$$

The unknown constants  $u_2, u_3, u_4$  can be determined by finite difference techniques by using the simultaneous equations

$$I_k(u_2, u_3, u_4) = \lambda_k I_1(u_2, u_3, u_4) \quad , \quad k = 2, 3, 4 \tag{5-19}$$

For example: let  $u_2^*, u_3^*$  and  $u_4^*$  be the solution of the above equation and take  $u_2^{(0)}, u_3^{(0)}$  and  $u_4^{(0)}$  as the initial values differing little from the exact solution  $u_2^*, u_3^*$  and  $u_4^*$ . We expand equation (5-19) in a Taylor's series in powers of the differences  $\alpha_k^{(1)} = u_k^* - u_k^{(0)}$  and terminate at the first order terms, thus obtaining the first approximation

## V. Propellant SLOSH Suppression

## 5.1 Anti-SLOSH Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

$$I_k^{(0)} + \sum_{j=2,3,4} \alpha_j^{(1)} I_{k,u_j}^{(0)} = \lambda_k \left[ I_1^{(0)} + \sum_{j=2,3,4} \alpha_j^{(1)} I_{1,u_j}^{(0)} \right] \quad k=2,3,4 \quad (5-20)$$

where

$$I_{k,u_j}^{(m)} = \left[ \frac{\partial I_k}{\partial u_j} \right]_{u_j = u_j^{(m)}}.$$

The partial derivatives which have limits not depending on  $u_2$ ,  $u_3$  and  $u_4$  may be obtained by direct differentiation, i.e., performing the partial differentiation, then integrating. Those that have  $u_2$ ,  $u_3$  and  $u_4$  as limits are computed numerically by using

$$I_{k,u_j}^{(m)} = \left[ I_k(u_j^{(m)} + \Delta u) - I_k^{(m)} \right] / \Delta u, \quad (5-21)$$

where  $\Delta u$  is a small increment. Solving for  $\alpha_j^{(1)}$  simultaneously from equation (5-20) and using the second approximation,

$$u_j^{(1)} = u_j^{(0)} + \alpha_j^{(1)}, \quad j=2, 3, 4$$

in equation (5-20), we obtain

$$I_k^{(1)} + \sum_{j=2,3,4} \alpha_j^{(2)} I_{k,u_j}^{(1)} = \lambda_k \left[ I_1^{(1)} + \sum_{j=2,3,4} \alpha_j^{(2)} I_{1,u_j}^{(1)} \right] \quad k=2,3,4.$$

The above process is carried out until  $u_k^{(m)}$ 's converge.

## V. Propellant Slosh Suppression

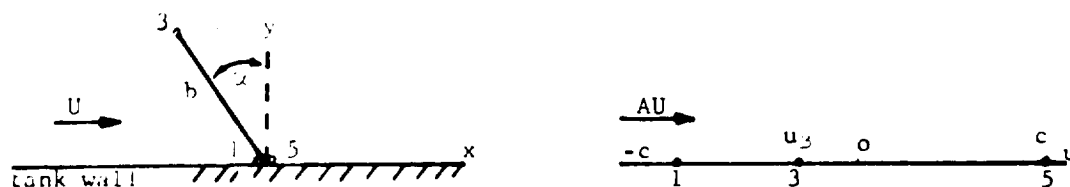
## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

Special Case: Baffle Type IA. This is a special case of Baffle Type I for which  $a = 0$ ,  $u_2 = -c$ , and  $u_4 = c$ . The transformation becomes

$$z = A \int_0^w (c+w)^{-\frac{(n+2)}{2n}} (w-u_3)(w-c)^{-\frac{(n-2)}{2n}} dw + B \quad (5-22)$$

The results are shown in the figure for Special Case Ia below.



Special Case Ia

For this case, the parameters and constants needed to obtain the pressure distribution (Equation 5-16) are the following:

$$K(u,0) = \frac{\sqrt{2}}{2} A (c+u)^{-\frac{(n+1)}{n}} (c-u)^{-\frac{(n-1)}{n}}$$

$$R(u,0) = \frac{\sqrt{2}}{2A} \left[ \frac{c+u}{c-u} \right]^{1/n} (u-u_3)^{-2}$$

$$f(u,0) = - \sqrt{2(c^2-u^2)} (u-u_3) \sin \alpha \quad (5-23)$$

$$g(u,0) = - \sqrt{2(c^2-u^2)} (u-u_3) \cos \alpha$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

$$A = b/I_2$$

$$B = b_1 + ib_2$$

$$\operatorname{Re} F(u, 0) = A \sin \alpha \int_0^u (c + u)^{-(n+2)/2n} (u - u_3) (c - u)^{-(n-2)/2n} du + b_1$$

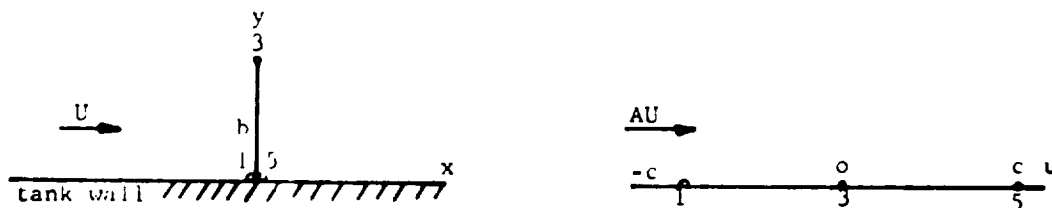
$$\operatorname{Im} F(u, 0) = -A \cos \alpha \int_0^u (c + u)^{-(n+2)/2n} (u - u_3) (c - u)^{-(n-2)/2n} du + b_2$$

The pressure distribution for Case IA is shown in figure (5-75).

Special Case: Baffle Type IB. This is a special case of Baffle Type I for which  $\alpha = 0$ ,  $u_2 = -c$ ,  $u_3 = 0$ ,  $u_4 = c$ , and  $\frac{1}{n} = 0$ . The transformation becomes

$$z = A \int_0^w \frac{wdw}{\sqrt{w^2 - c^2}} = A \sqrt{w^2 - c^2}. \quad (5-24)$$

The results are shown in the figure for special Case IB below.



Special Case Ib

The pressure distribution is well known. It is parabolic in shape with the maximum disturbance pressure occurring at the base of the baffle and the minimum (zero) at the tip of the baffle.

## V. Propellant Slosh Suppression

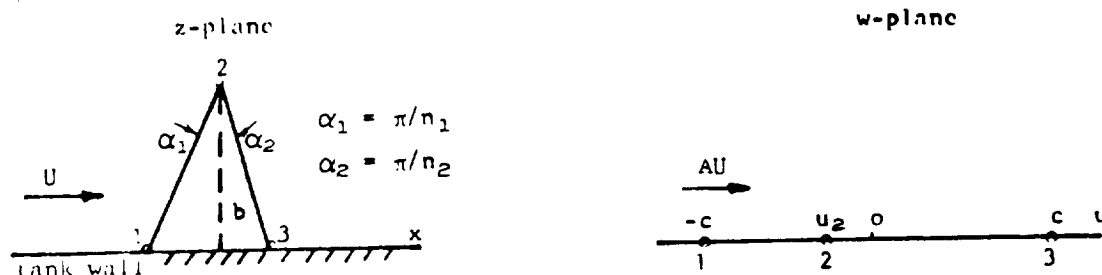
## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

Baffle Type II. The baffle-tank system ( $n_1 = n_2 = n$ ,  $u_2 = 0$ ) in the  $z$ -plane is mapped onto the  $w$ -plane by the transformation

$$z = A \int_0^w (c+w)^{-\frac{(n_1-2)}{2n_1}} (w-u_2)^{1-\frac{1}{n_1}-\frac{1}{n_2}} (w-c)^{-\frac{(n_2-2)}{2n_2}} dw + B \quad (5-25)$$

The results of this transformation is shown in the figure below.

**Baffle Type II**

The parameters and constants given in Equation (5-16) can be obtained as follows:

$$\frac{dz}{dw} = AK' (\sin \alpha_1 + i \cos \alpha_1) = K(f + ig)$$

$$\frac{dw}{dz} = \frac{1}{AK'} (\sin \alpha_1 - i \cos \alpha_1) = R(f - ig) \quad (5-26)$$

$$-c \leq u \leq u_2$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

$$\frac{dz}{dw} = AK' (\sin \alpha_2 - i \cos \alpha_2) = K(f + ig)$$

$$\frac{dw}{dz} = \frac{1}{AK'} (\sin \alpha_2 + i \cos \alpha_2) = R(f - ig) \quad (5-27)$$

$$-u_2 \leq u \leq c$$

$$K' = (c+u)^{-\frac{(n_2-2)}{2n_2}} |u-u_2|^{1-\frac{1}{n_1}-\frac{1}{n_2}} (c-u)^{-\frac{(n_2-2)}{2n_2}}$$

$$A = \frac{b}{I_1 \cos \alpha_1} \quad I_1 = \int_{-c}^{u_2} K' du$$

$$b_1 = b \left( \tan \alpha_2 - \frac{\sin \alpha_2}{\cos \alpha_1} \frac{I_c}{I_1} \right), \quad I_c = \int_0^c K' du$$

$$b_2 = b \left( \frac{\cos \alpha_2}{\cos \alpha_1} \frac{I_c}{I_1} \right) \quad (5-28)$$

The constant  $u_2$  can be found from

$$\frac{\cos \alpha_1}{\cos \alpha_2} I_1 = I_c - I_2, \quad I_2 = \int_0^{u_2} K' du$$



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

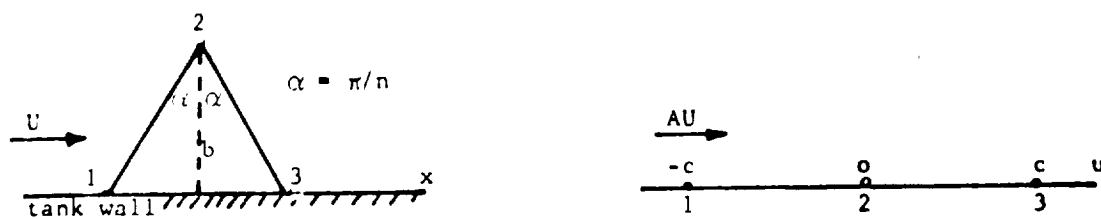
## 5.1.7.3 Procedure III (83) (continued)

Special Case: Baffle Type IIA. Same as Baffle Type II

but  $\alpha = \alpha_1 = \alpha_2$ . The transformation becomes

$$z = A \int_0^w \left[ (w^2 - c^2)^{-\frac{(n-2)}{2n}} (w)^{\frac{(n-2)}{n}} \right] dw + ib \quad (5-29)$$

The results are shown in the figure below



Baffle Type IIA

Baffle Type III. The transformation from the  $z$ -plane

to the  $w$ -plane is given by

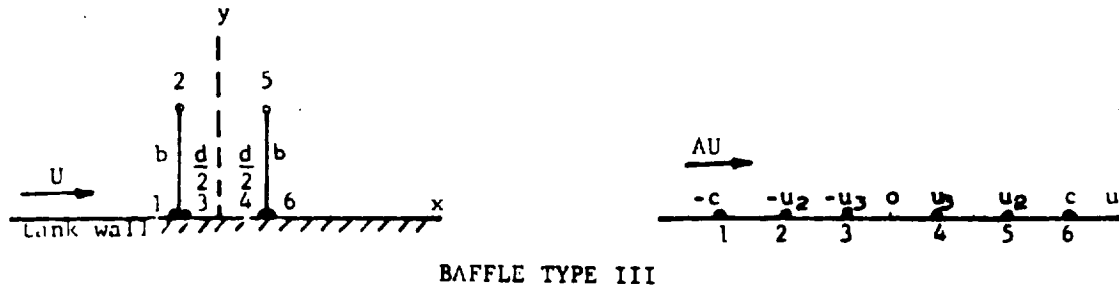
$$z = A \int_0^w \frac{(w^2 - u_2^2) dw}{\sqrt{(w^2 - c^2)(w^2 - u_3^2)}} \quad (5-30)$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.3 Procedure III (83)(continued)

and the results shown in the figure below.



The parameters and constants given in Equation (5-16) are as follows:

$$\frac{dz}{dw} = A \frac{(u^2 - u_2^2)}{\sqrt{(c^2 - u^2)(u^2 - u_3^2)}} (o - i) = K(f + ig) \quad (5-31)$$

$$\frac{dw}{dz} = \frac{1}{A} \frac{\sqrt{(c^2 - u^2)(u^2 - u_3^2)}}{(u^2 - u_2^2)} (o + i) = R(f - ig)$$

$$\text{for } u_3 \leq |u| \leq c$$

$$\frac{dz}{dw} = A \frac{(u^2 - u_2^2)}{\sqrt{(c^2 - u^2)(u_3^2 - u^2)}} (1 + io) = K(f + ig) \quad (5-32)$$

$$\frac{dw}{dz} = \frac{1}{A} \frac{\sqrt{(c^2 - u^2)(u_3^2 - u^2)}}{u^2 - u_2^2} (1 - io) = R(f - ig)$$

$$\text{for } |u| \leq u_3$$

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

## 5.1.7.3 Procedure III (83) (continued)

The constants  $u_2$ ,  $u_3$ , and  $A$  are determined simultaneously from the equations

$$K_1/K_2 = d/2b, \quad K_2 = K_3, \quad A = d/2K_1, \quad B = 0$$

$$K_1 = \int_0^{u_3} \frac{(u_2^2 - u^2) du}{\sqrt{(c^2 - u^2)(u_3^2 - u^2)}}$$

$$K_2 = \int_{u_3}^{u_2} \frac{(u_2^2 - u^2) du}{\sqrt{(c^2 - u^2)(u_3^2 - u^2)}} \quad (5-33)$$

$$K_3 = \int_{u_2}^c \frac{(u^2 - u_2^2) du}{\sqrt{(c^2 - u^2)(u^2 - u_3^2)}}$$

Special Case: Baffle Type IIIA. The results are identical with Special Case Baffle Type IB but with  $u_3 = u_2 = 0$ .

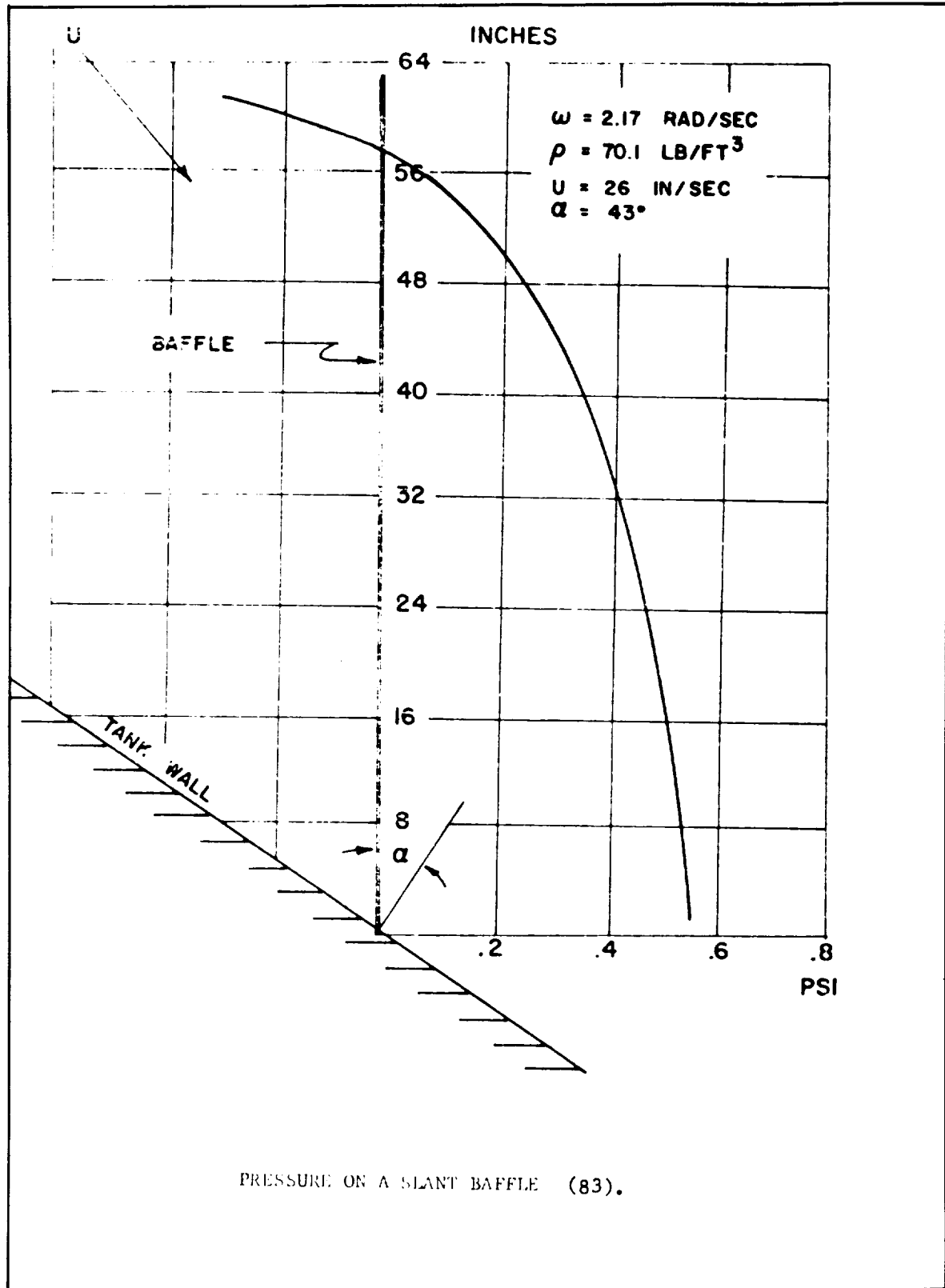


Figure 5-75

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

5.1.8 Conclusions and Recommendations

In choosing a damping system, the designer should start with solid ring baffles and use Miles' expression, Equation (5-1), for his first approximation of the damping factor. The reason is that Miles' equation is simple in format and in nature and a rather rapid "ball-park figure" for the spacing and width of the baffles can be obtained. Bauer's extension (article 5.1.3.2.3) would then give more accurate results and optimum conditions. The pressure acting on the baffle (Equations 5-10, 5-11, and 5-16) will permit the optimization of the thickness and material of each baffle and thus the calculation of the total weight added by the damping system.

Results

1. The maximum frequency occurs when the baffle is located at the undisturbed free liquid surface, i.e.,  $\frac{d_s}{R} = 0$  (Figures 5-2 and 5-3).  
Note: The curves in Figures 5-2 and 5-3, for high values of  $\frac{d_s}{R}$ , approach asymptotically, the first resonant frequency for a circular cylindrical tank containing no baffles.
2. Agreement between experimental values of the damping factor and those obtained from Miles' equation is good except in the range of  $0 < \frac{d_s}{R} < 0.125$  (Figure 5-5).
3. With the hole size constant and for a given baffle depth:
  - (a) The frequency increases for increasing percent perforation (Figure 5-2).

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

Results (continued)

- (b) The damping factor increases for decreasing hole size  
(Figure 5-8).
- 4. With the percent perforation constant and for a given baffle depth:
  - (a) The frequency increases for increasing hole size  
(Figure 5-3).
  - (b) The damping factor increases for decreasing hole size  
(Figure 5-7).
- 5. The frequency of a solid fixed ring baffle is higher than that of a perforated baffle for  $\frac{d_s}{R} \leq 0.05$  and lower for  $\frac{d_s}{R} \geq 0.05$   
(Figures 5-2 and 5-3).
- 6. Damping produced by perforating baffles is consistently lower than that produced by a solid baffle (Figure 5-7 and 5-8).
- 7. The damping factor increases as the excitation amplitude increases  
(Figure 5-6).
- 8. For a given baffle depth:
  - (a) The damping factor of a fixed ring, conic section and perforated conic section increases as the baffle width increases  
(Figures 5-10, 5-67, and 5-68).
  - (b) With the radial clearance constant, the damping factor of a ring-with-radial-clearance increases as its baffle width increases (Figure 5-11).
  - (c) With the baffle width constant, the damping factor of a ring-with-radial-clearance decreases as the radial clearance increases (Figure 5-11).

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

9. The damping factor is independent of kinematic viscosity and increases as the excitation amplitude increases (Figures 5-13 and 5-14).
10. Agreement between experimental values of the damping factor and those obtained in Miles' equation is good except for  $0 < \frac{d_s}{R} \leq (0.8 - 1.2)$ , (Figures 5-12 and 5-14).
11. For a given baffle depth:
  - (a) The frequency for a fixed ring, conic section, and perforated conic section increases as the baffle width decreases (Figures 5-15, 5-64, and 5-66).
  - (b) With the radial clearance constant, the frequency increases as the baffle width decreases (Figure 5-16).
  - (c) With the baffle width constant, the frequency increases as the radial clearance decreases (Figure 5-16).
12. During propellant loading, the cans erected smoothly and arranged themselves in the required configuration even though they were randomly placed in the tank bottom before starting the filling operation. For this reason, they operated as good anti-vortex devices.
13. The devices floated perfectly and yielded high damping effects.
14. Draining the liquid while the tank was oscillating did not change the damping effect.
15. An inherent disadvantage of cans is that they are free to move when the container is empty, i.e., they are loose bodies during the transportation of the missile and can thereby cause damage to the inner tank structure.

## V. Propellant SLOSH Suppression

## 5.1 Anti-SLOSH Devices (continued)

16. The dashed parts of the curves in Figures 5-9, 5-10, and 5-67 indicate that, because of the turbulent nature of the liquid in the regions of higher damping, an uncertainty existed in defining the exact location of maximum damping.
17. A divided baffle which extended down to the ring baffle was used for most of the tests in Reference (35). This tended to increase the value of the damping as seen in Figure 5-14.

Conclusions

1. Fixed ring baffles can be perforated with holes of relatively small diameter, thereby reducing the baffle area by as much as 23% with no appreciable loss in damping effectiveness. Such reduction in baffle area may be effectively used for additional baffles thus increasing the minimum damping ratio value with no appreciable increase in baffle weight.
2. For an excitation amplitude  $\frac{x_o}{d} = 0.00417$ , the baffle above the liquid surface adds to damping only when the distance  $\frac{d_s}{R} < 0.125$  and the baffle below the surface contributes to damping only when  $0 \leq \frac{d_s}{R} \leq 0.375$ . The baffle spacing for some minimum  $\gamma_s$  can be determined by adding the depth of the submerged baffle to the depth of  $\frac{d_s}{R} = 0.125$  for which the upper baffle is still effective.
3. Fixed ring baffles can efficiently introduce liquid damping into propellant tanks.
4. Fixed ring baffles are not very efficient for shifting the liquid resonant frequency.



## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

Conclusions (continued)

5. Based on the total surface area of a baffle, the highest mean damping factor of the baffles tested appears to be afforded by the fixed ring baffle (Figure 5-78).

Summarizing, the following remarks are pertinent.

1. Bauer's extension to Miles' equation can be used without further experimental verification except for a check of the final product.
2. Ring baffles provide little frequency effect.
3. Solid ring baffles with small diameter perforations (about 20% open area) can effectively be used to reduce baffle weight addition without an appreciable loss in damping effectiveness.
4. Conical rings offer greater degree of damping than floating cans, except where the cans carefully cover the liquid surface.
5. The amount of damping produced by both floating cans and conical rings is strongly dependent upon excitation amplitude.
6. Cans are the only suitable type of float. However, ring baffles are more reliable and appear to give greater damping.
7. Floats, like baffles, are used mostly to damp liquid motion; however, floats are like partitions, in that their effect on frequency and damping is constant for all shallow depths.
8. Subdivision by radial walls is much more effective in reducing the amount of sloshing than subdivision by concentric walls. In the case of a quarter-sectored tank the vibrating liquid mass is reduced to more than  $1/2$  that of a circular cylinder.
9. Ring baffles are far better as damping devices than conic, inverted conic or cruciform baffles.

## V. Propellant Slosh Suppression

## 5.1 Anti-Slosh Devices (continued)

10. Cross partitions have a greater effect on the eigen-frequencies than do concentric partitions.

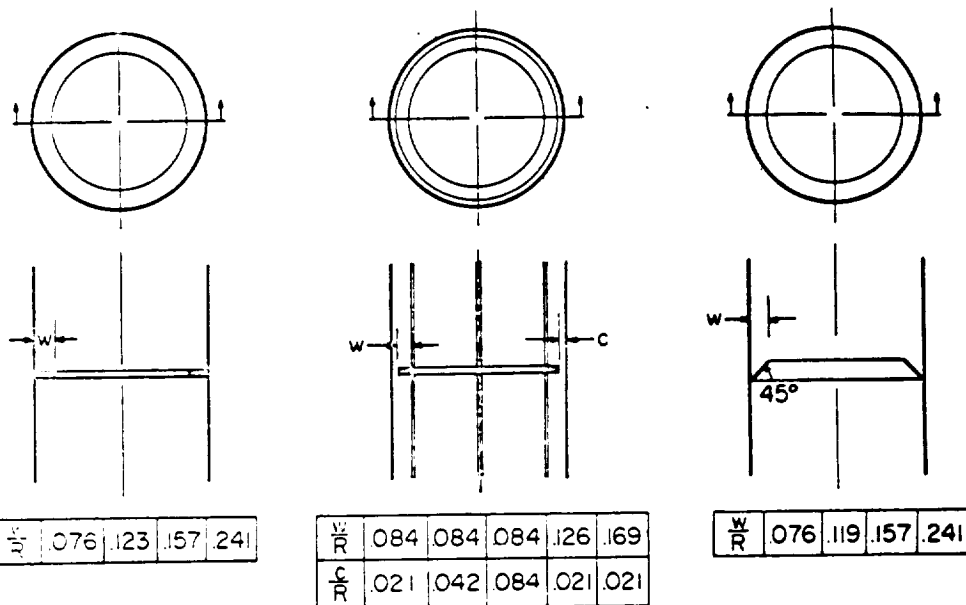
11. Floats should cover the entire cross-sectional area and fill the zone of oscillation, i.e., to a depth of one-fourth the tank diameter.

12. Floats should be capable of following the changing liquid level.

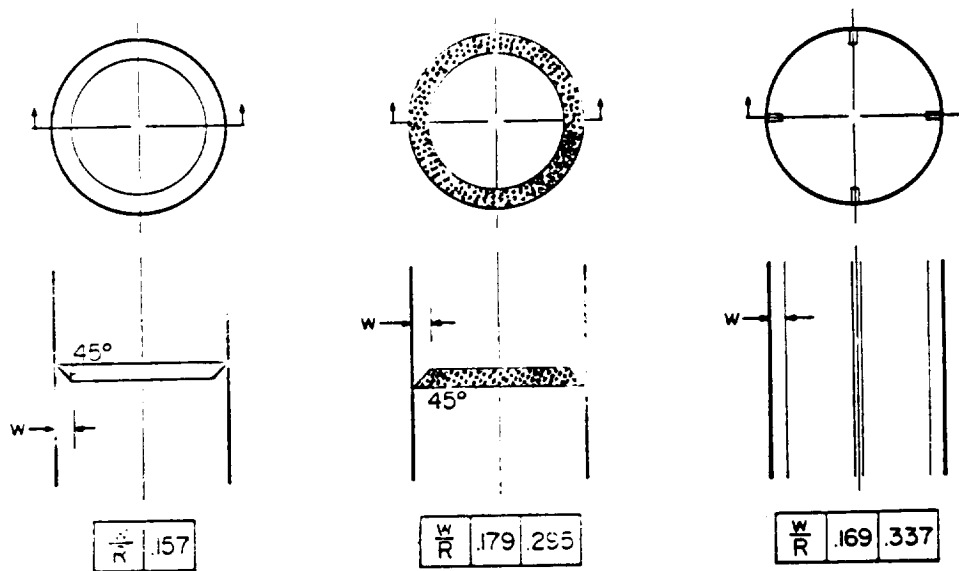
13. Floats should be capable of adapting to the changing cross-sectional area of the container, i.e., the float should not catch-on or cling to stiffener rings, pipelines or other equipment obstructing the inner walls of the tank.

14. Floats should not change the moment of inertia of the liquid when subjected to roll excitations about the longitudinal axis of the container.

15. Partitions should divide the total liquid mass into approximately equal partial masses thereby effectively reducing the tank diameter.



(a) Fixed ring. (b) Ring with radial clearance. (c) Conic section.



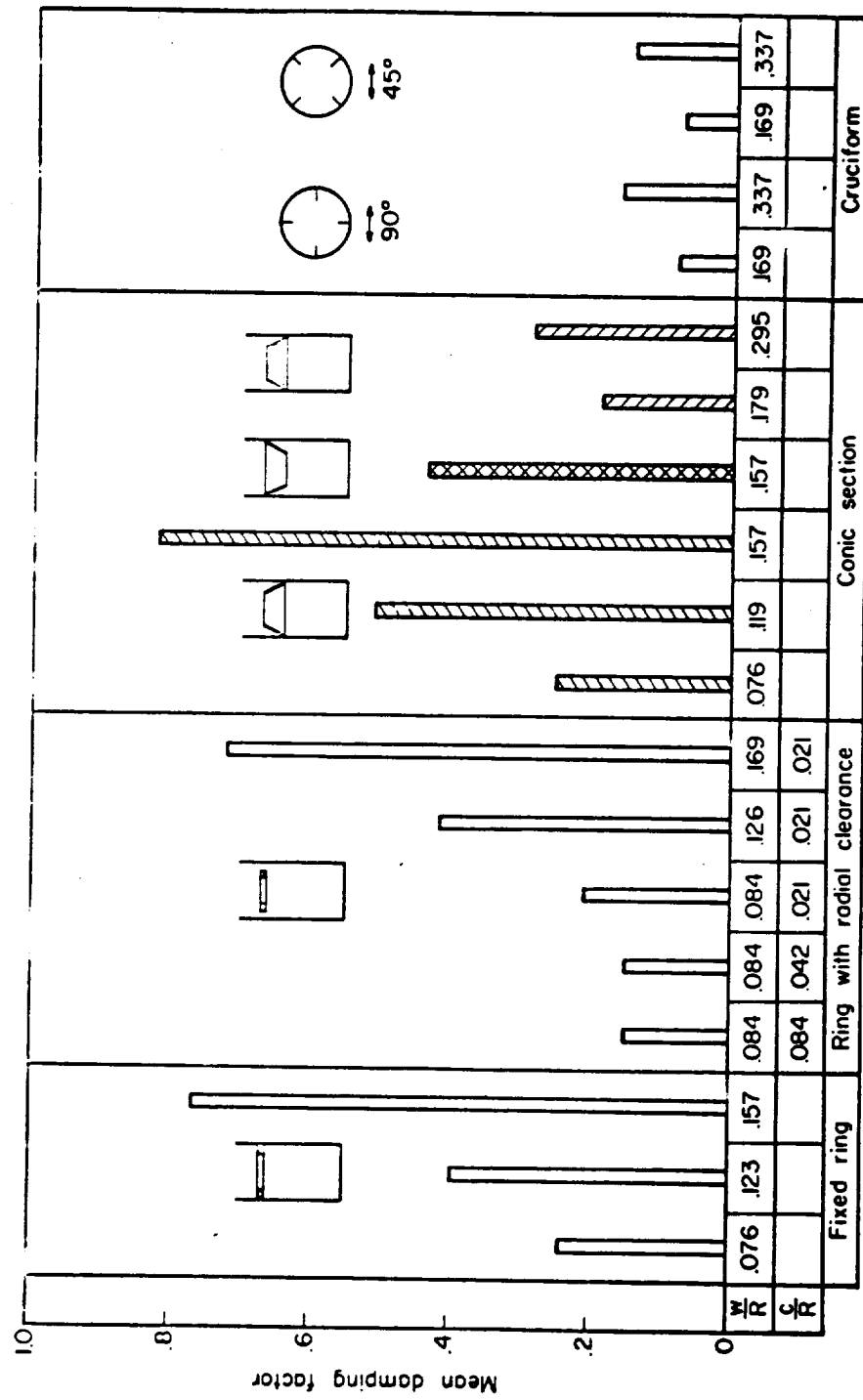
(d) Inverted conic section.

(e) Perforated conic section.

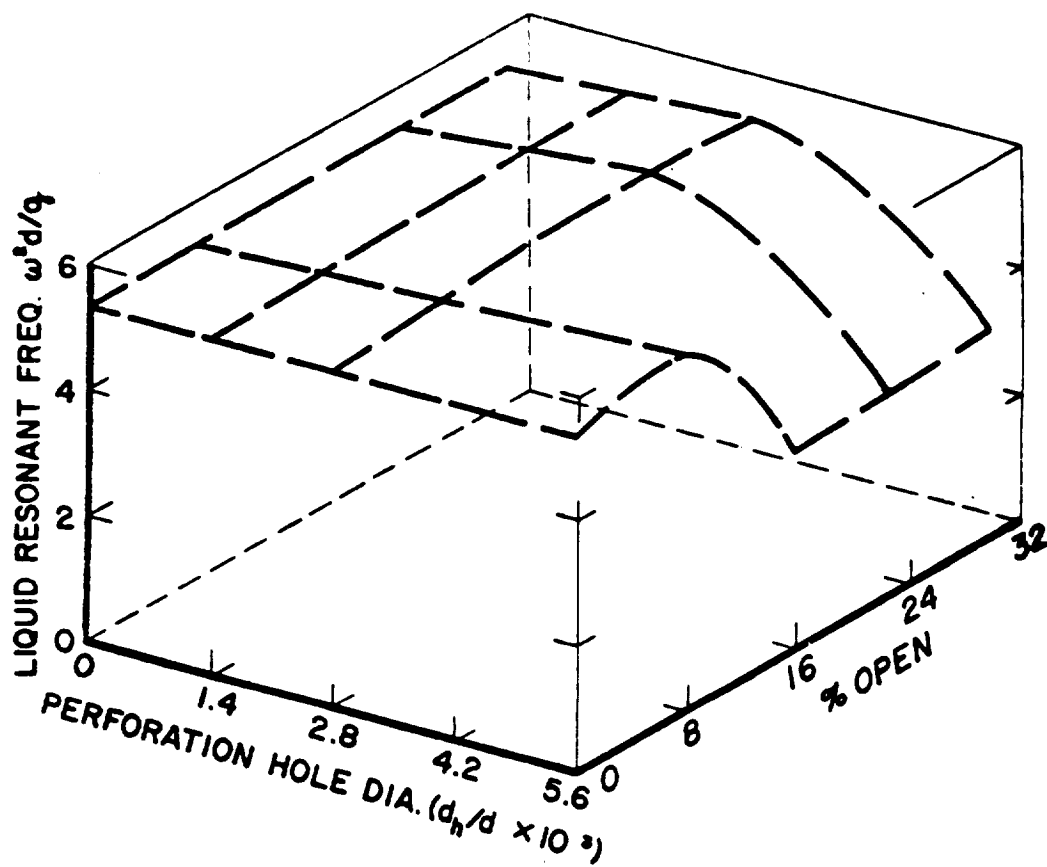
(f) Cruciform.

Baffle configurations. (35)

Figure 5-76



Comparison of baffles. (35)



Effect of hole size and percent perforation  
on resonant frequency for a quarter sector tank. (58)

Figure 5-78

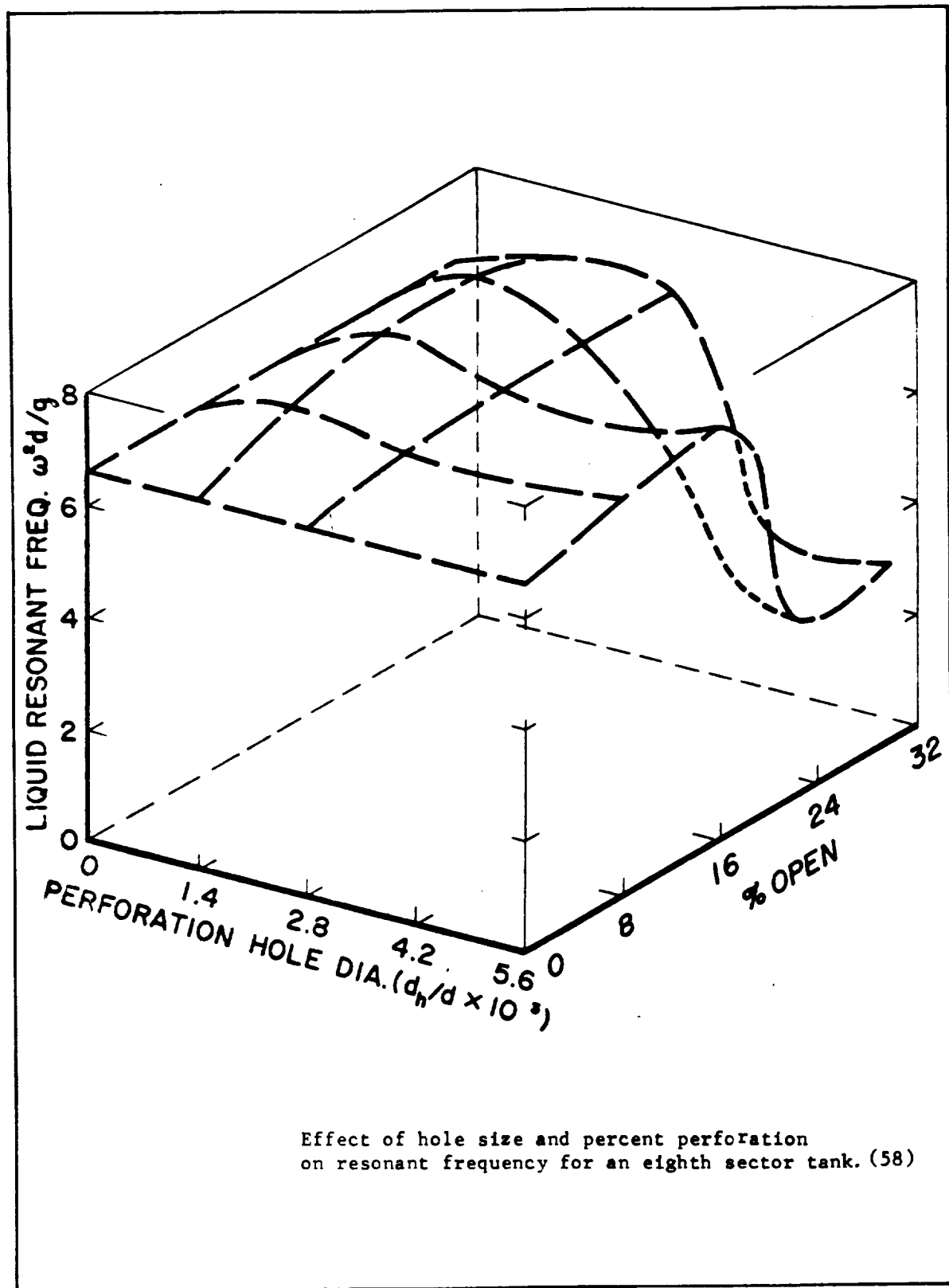


Figure 5-79

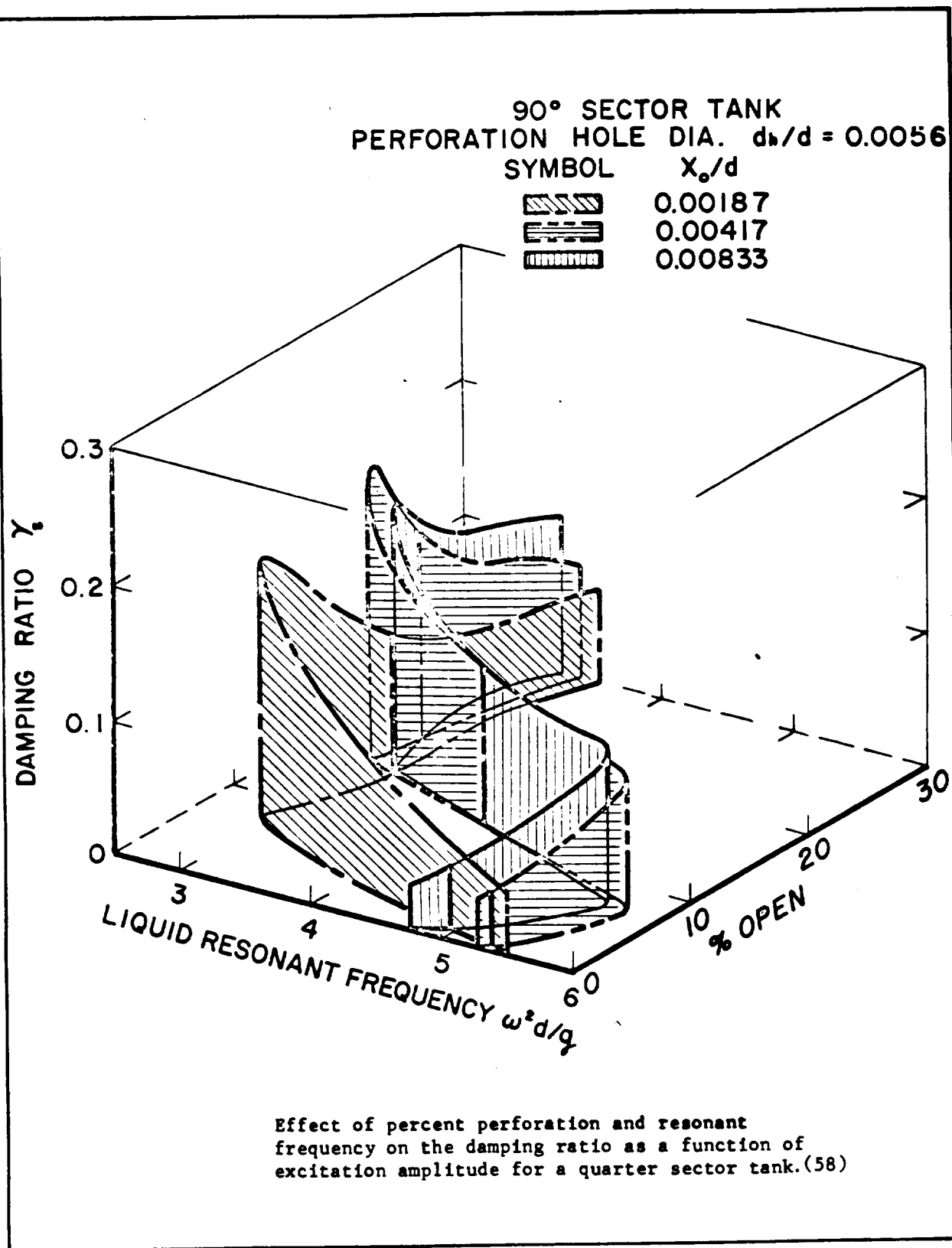


Figure 5-80

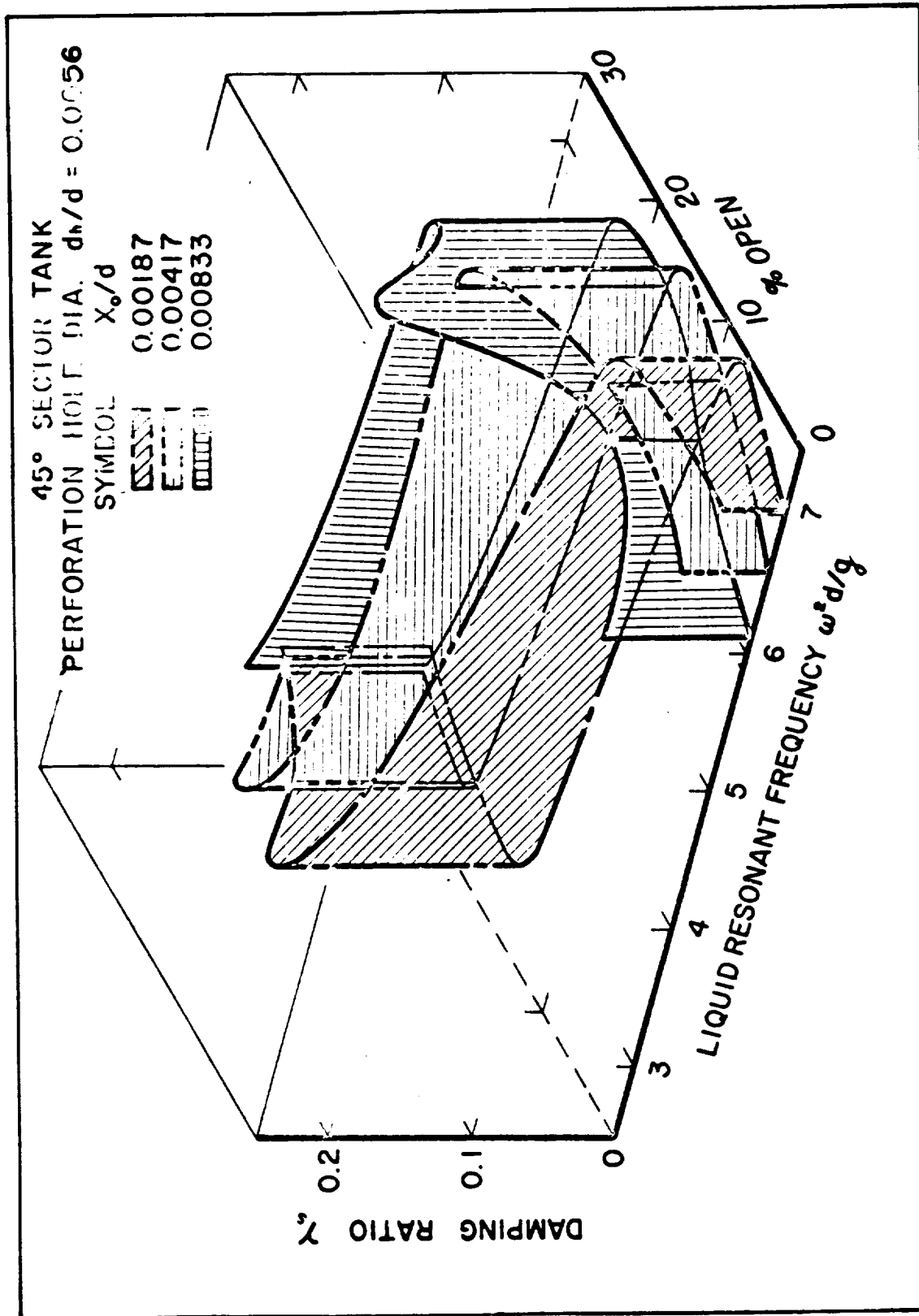


Figure 5-81. Effect of percent perforation and resonant frequency on the damping ratio as a function of excitation amplitude for an eighth sector tank. (58)



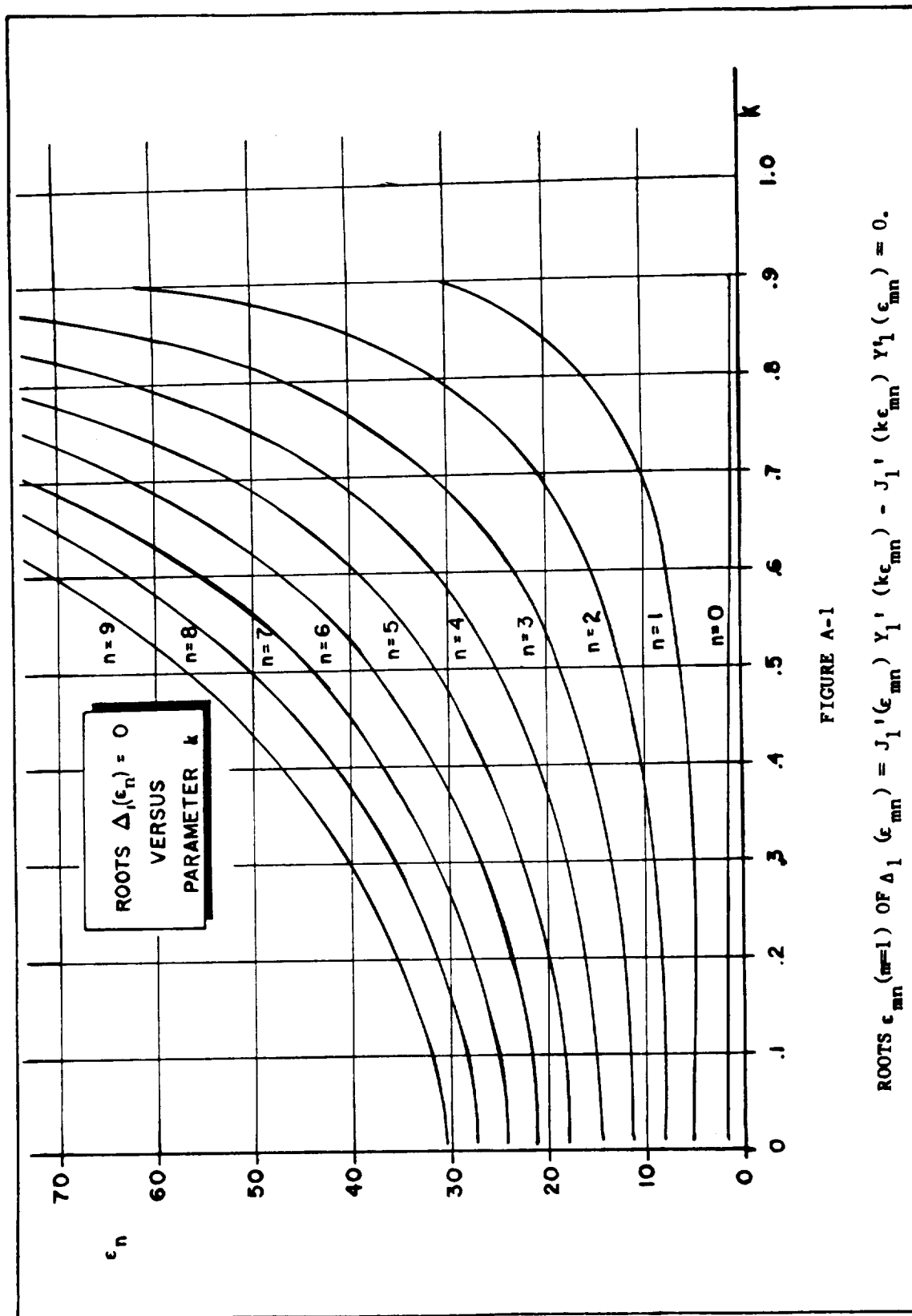


FIGURE A-1

ROOTS  $\epsilon_{mn} (\alpha=1)$  OF  $\Delta_1(\epsilon_{mn}) = J_1'(\epsilon_{mn}) Y_1'(k\epsilon_{mn}) - J_1'(k\epsilon_{mn}) Y_1'(\epsilon_{mn}) \approx 0$ .

TABLE A-1  
CONTAINER VELOCITY POTENTIAL FOR CIRCULAR CYLINDRICAL TANK OR DERIVATIVE THEREOF.

| Excitation   | $\phi_1$ - Container Velocity Potential  |
|--|--|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$      | $\phi_1 = i\omega r x_0 e^{i\omega t} \cos\phi = i\omega r x \cos\phi$           |
| Translation in the y-direction, $y = y_0 e^{i\omega t}$      | $\phi_1 = i\omega r y_0 e^{i\omega t} \sin\phi = i\omega r y \sin\phi$           |
| Pitching about the x-axis, $\chi = \chi_0 e^{i\omega t}$     | $\phi_1 = i\omega z \chi_0 e^{i\omega t} \sin\phi = i\omega z \chi \sin\phi$     |
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$ | $\phi_1 = i\omega z \theta_0 e^{i\omega t} \cos\phi = i\omega z \theta \cos\phi$ |
| Roll about the z-axis, $\phi = \phi_0 e^{i\omega t}$         | $\phi_1 = i\omega r^2 \phi_0 e^{i\omega t} = i\omega r^2 \phi$                   |

TABLE A-2  
CONTAINER VELOCITY POTENTIAL FOR RECTANGULAR TANK.

| Excitation  | $\phi_1$ - Container Velocity Potential     |
|---|---|
| Translation in the x-direction, $x = x_0 \sin\omega t$      | $\phi_1 = \omega x x_0 \sin\omega t$        |
| Pitching about the y-axis, $\theta = \theta_0 \sin\omega t$ | $\phi_1 = \omega x z \theta_0 \cos\omega t$ |
| Roll about the z-axis, $\phi = \phi_0 \sin\omega t$         | $\phi_1 = \omega x y \phi_0 \cos\omega t$   |

TABLE A-3

ZEROS  $\epsilon_{mn}$  ( $m=1$ ) OF THE FIRST DERIVATIVE OF THE BESSEL  
 FUNCTION OF THE FIRST ORDER AND FIRST KIND [ $J'_1(\epsilon_{mn}) = 0$ ].

| $n$ | $\epsilon_n$ | $n$ | $\epsilon_n$ | $n$ | $\epsilon_n$ |
|-----|--------------|-----|--------------|-----|--------------|
| 0   | 1.84119      | 11  | 36.89000     | 22  | 71.45677     |
| 1   | 5.33144      | 12  | 40.03341     | 23  | 74.60000     |
| 2   | 8.53631      | 13  | 43.17665     | 24  | 77.74383     |
| 3   | 11.70600     | 14  | 46.31957     | 25  | 80.88514     |
| 4   | 14.86359     | 15  | 49.46238     | 26  | 84.02718     |
| 5   | 18.01553     | 16  | 52.60507     | 27  | 87.16914     |
| 6   | 21.16440     | 17  | 55.74758     | 28  | 90.31228     |
| 7   | 24.31145     | 18  | 58.89000     | 29  | 93.45301     |
| 8   | 27.45751     | 19  | 62.03234     | 30  | 96.59499     |
| 9   | 30.60194     | 20  | 65.17465     | 31  | 99.73675     |
| 10  | 33.74397     | 21  | 68.31682     |     |              |

TABLE A-4  
 ROOTS  $\epsilon_{mn}$  OF THE FIRST DERIVATIVE OF THE BESSEL  
 FUNCTION OF ORDER  $2m$  AND OF THE FIRST KIND [ $J'_{2m}(\epsilon_{mn}) = 0$ ].

| $\frac{m}{n}$ | 0      | 1      | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      |
|---------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0             | 3.832  | 3.054  | 5.318  | 8.105  | 9.648  | 11.716 | 13.881 | 15.917 | 18.104 | 20.189 |
| 1             | 7.016  | 6.706  | 9.282  | 11.735 | 14.115 | 16.448 | 18.745 | 21.015 | 23.261 | 25.495 |
| 2             | 10.173 | 9.969  | 12.682 | 15.268 | 17.774 | 20.223 | 22.629 | 25.002 | 27.347 | 29.670 |
| 3             | 13.324 | 13.170 | 15.964 | 18.637 | 21.229 | 23.761 | 26.246 | 28.694 | 31.112 | 33.504 |
| 4             | 16.471 | 16.348 | 19.196 | 21.932 | 24.587 | 27.182 | 29.739 | 32.237 | 34.742 | 37.160 |
| 5             | 19.616 | 19.513 | 22.401 | 25.184 | 27.889 | 30.535 | 33.131 | 35.689 | 38.212 | 40.707 |
| 6             | 22.760 | 22.672 | 25.590 | 28.110 | 31.155 | 33.842 | 36.481 | 39.079 | 41.643 | 44.178 |
| 7             | 25.904 | 25.826 | 28.768 | 31.618 | 34.397 | 37.118 | 39.792 | 42.426 | 45.052 | 47.595 |
| 8             | 29.047 | 28.978 | 31.939 | 34.813 | 37.620 | 40.371 | 43.075 | 45.740 | 48.371 | 50.971 |
| 9             | 32.189 | 32.127 | 35.104 | 38.000 | 40.830 | 43.607 | 46.338 | 49.030 | 51.687 | 54.315 |

TABLE A-5  
 ROOTS  $\epsilon_{mn}$  OF THE FIRST DERIVATIVE OF THE BESSEL FUNCTION  
 OF ORDER  $\epsilon_m$  AND OF THE FIRST KIND  $[J'_{\epsilon_m}(\epsilon_{mn}) = 0]$ .

|   | 1        | 3        | 5        |
|---|----------|----------|----------|
| 0 | 5.31756  | 13.82109 | 22.26759 |
| 1 | 9.28240  | 18.74485 | 27.71172 |
| 2 | 12.68191 | 22.62927 | 31.97366 |
| 3 | 15.96411 | 26.24604 | 35.87393 |
| 4 | 19.19603 | 29.72898 | 39.58453 |
| 5 | 22.40103 | 33.13145 | 43.17654 |

TABLE A-6

ROOTS  $\epsilon_{mn}$  ( $m=1$ ) OF THE DETERMINANT

$$\Delta_1(\epsilon_{mn}) = J_1'(\epsilon_{mn}) Y_1'(k\epsilon_{mn}) - J_1'(k\epsilon_{mn}) Y_1'(\epsilon_{mn}) = 0$$

WHERE  $J_1'$  AND  $Y_1'$  ARE THE FIRST DERIVATIVES OF

THE BESSEL FUNCTIONS OF ORDER 1 AND OF

THE FIRST AND SECOND KIND RESPECTIVELY.

| $\frac{k}{n}$ | 0       | .1      | .2      | .3      | .4      | .5      | .6      | .7      | .8       | .9       |
|---------------|---------|---------|---------|---------|---------|---------|---------|---------|----------|----------|
| 0             | 1.8412  | 1.8035  | 1.7051  | 1.5821  | 1.4618  | 1.3547  | 1.2621  | 1.2824  | 1.1134   | 1.0532   |
| 1             | 5.3314  | 5.1371  | 4.9609  | 5.1374  | 5.6592  | 6.5649  | 8.0411  | 10.5919 | 15.7781  | 31.4470  |
| 2             | 8.5363  | 8.1992  | 8.4331  | 9.3084  | 10.6834 | 12.7066 | 15.8013 | 21.0041 | 31.4513  | 62.8481  |
| 3             | 11.7060 | 11.3588 | 12.1650 | 13.6837 | 15.8481 | 18.9427 | 23.6239 | 31.4558 | 47.1504  | 94.2645  |
| 4             | 14.8636 | 14.6344 | 15.9932 | 18.1159 | 21.0488 | 25.2025 | 31.4632 | 41.9190 | 62.8510  | 125.6750 |
| 5             | 18.0155 | 17.9864 | 19.8616 | 22.5707 | 26.2641 | 31.4721 | 39.3076 | 52.3845 | 78.5549  | 157.0880 |
| 6             | 21.1644 | 21.3837 | 23.7502 | 27.0369 | 31.4859 | 37.7459 | 47.1552 | 62.8523 | 94.2601  | 188.5022 |
| 7             | 24.3114 | 24.8081 | 27.6498 | 31.5092 | 36.7119 | 44.0223 | 55.0047 | 73.3212 | 109.9662 | 219.9170 |
| 8             | 27.4575 | 28.2497 | 31.5563 | 35.9855 | 41.9403 | 50.3005 | 62.8553 | 83.7910 | 125.6728 | 251.3320 |
| 9             | 30.6019 | 31.7027 | 35.4675 | 40.4643 | 47.1704 | 56.5797 | 70.7066 | 94.2612 | 141.3797 | 282.7473 |

Table A-7. Liquid Displacements

| Circular Cylindrical Tank   | Excitation: Harmonic Translation and/or Pitching |
|---|--|
| <p>Translation in the x-direction, <math>x = x_0 e^{i\omega t}</math></p> $\zeta = \frac{\omega^2}{g} x_0 e^{i\omega t} \cos \phi \left[ r + \sum_{n=0}^{\infty} \frac{2a J_1(\rho_n)}{(\epsilon_n^2 - 1)(\eta_n^2 - 1) J_1(\epsilon_n)} \right]$ <p>Pitching about the y-axis, <math>\theta = \theta_0 e^{i\omega t}</math></p> $\zeta = -\omega^2 \theta_0 e^{i\omega t} \cos \phi \left\{ \frac{hr}{2g} + \sum_{n=0}^{\infty} \frac{2a J_1(\rho_n)}{(\epsilon_n^2 - 1)(\eta_n^2 - 1) J_1(\epsilon_n)} \left[ \frac{2}{\omega^2 \cosh \kappa_n} + \frac{h}{2g} - \frac{1}{\omega^2} \right] \right\}$ |  |

Table A-8. Liquid Displacements

| Quarter-Sector Tank   | Excitation: Harmonic Translation and/or Pitching |
|---|--|
| Translation in the x-direction, $x = x_0 e^{i\omega t}$   |  |
| $\zeta = \frac{\omega^2}{g} x_0 e^{i\omega t} \left[ r \cos \phi + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\bar{a}_m \bar{b}_n J_{2m}(\rho_{mn}) \cos 2m\phi}{(\eta_{mn}^2 - 1)} \right]$   |  |
| Pitching about the y-axis, $\theta = \theta_0 e^{i\omega t}$  |  |
| $\zeta = -\frac{\omega^2}{g} \theta_0 e^{i\omega t} \left\{ \frac{hr}{2} \cos \phi - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \bar{A}_{mn} \cosh \left( \frac{\kappa_{mn}}{2} \right) + \bar{B}_{mn} \sinh \left( \frac{\kappa_{mn}}{2} \right) J_{2m}(\rho_{mn}) \cos 2m\phi \right] \right\}$ |  |



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